The operator product expansion for perturbative quantum field theory in curved spacetime

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Abstract

We present an algorithm for constructing the Wilson operator product expansion (OPE) for perturbative interacting quantum field theory in general Lorentzian curved spacetimes, to arbitrary orders in perturbation theory. The remainder in this expansion is shown to go to zero at short distances in the sense of expectation values in arbitrary Hadamard states. We also establish a number of general properties of the OPE coefficients: (a) they only depend (locally and covariantly) upon the spacetime metric and coupling constants, (b) they satsify an associativity property, (c) they satisfy a renormalization group equation, (d) they satisfy a certain microlocal wave front set condition, (e) they possess a "scaling expansion". The latter means that each OPE coefficient can be written as a sum of terms, each of which is the product of a curvature polynomial at a spacetime point, times a Lorentz invariant Minkowski distribution in the tangent space of that point. The algorithm is illustrated in an example.

1 Introduction

The operator product expansion [38] (OPE, for short) states that a product of n local quantum fields can be expanded at short distances as an asymptotic series, each term of which is given by a model dependent coefficient function of the n spacetime arguments, times a local field at a nearby reference point[1] y:

$$\mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\cdots\mathcal{O}_{i_n}(x_n) \sim \sum_k C_{i_1 i_2 \ldots i_n}{}^k(x_1, x_2, \ldots, x_n)\, \mathcal{O}_k(y)\,. \tag{1}$$

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[1] In this paper, we will take $y = x_n$ but other more symmetric choices are also possible.

This expansion has been established in perturbative quantum field theory on Minkowski spacetime [40], and is by now a standard tool, for example in the analysis of quantum gauge theories such as QCD. It has also been proven for conformally invariant quantum field theories [36, 32, 33], and has in fact played a major role in the developement and analysis of such theories [6, 28]. Formal mathematical proofs of the OPE have also been given within various axiomatic settings [15, 2] for quantum field theory on Minkowski spacetime. Given the importance of the OPE in flat spacetime, it is of great interest to construct a corresponding version of the expansion in curved spacetime.

In this paper, we present such a construction, within the framework of perturbation theory. It is based on the perturbative construction of quantum field theories on globally hyperbolic Lorentzian spacetimes which was recently achieved in a series of papers [21, 22, 23, 24], which in turn were based upon key results of Brunetti, Fredenhagen, and Köhler [4, 3]. In these papers, the interacting (Heisenberg) quantum fields in curved spacetime are constructed as formal power series in the coupling constant(s), that are valued in a certain *-algebra of quantum observables. The basic idea how to construct the OPE for these interacting fields is as follows: Suppose we have linear functionals Ψ_y^k from the algebra into the complex numbers, that are labelled by an index k enumerating the different composite fields, and a reference spacetime point y, and which form a "dual basis" to the composite quantum fields in the sense that $\Psi_y^k(\mathcal{O}_j(y)) = \delta^k{}_j$. Now apply the functional with label k to the OPE. Then we immediately see that the operator product coefficient in front of the k-th term in the sum on the right side of the OPE ought to be given precisely by the c-number distribution obtained by applying that functional to the product of fields on the left side of the OPE. Below, we will give a perturbative construction of such a dual basis of functionals in the context of a scalar, renormalizable field theory model in any 4 dimensional Lorentzian spacetime. In this way, we will obtain the desired perturbative formula for the OPE coefficients, C.

While this construction gives a conceptually clean derivation of a perturbative expression for the coefficients C, it does not yet show that the remainder of the OPE expansion defined in this way actually goes to zero (and in what sense) when the points are scaled together. To analyze this question, we apply any Hadamard state to the remainder in the OPE. We then show that the resulting c-number distribution in n points goes to zero in the sense of distributions when the points are scaled together in a arbitrary fashion. Thus, the OPE holds in the sense of an asymptotic expansion of expectation values, to arbitrary order in perturbation theory, and for any Hadamard state. The proof of this statement mainly relies on the known scaling properties of the various terms in the perturbative series for the interacting fields [21, 22, 4]. However, these properties by themselves are not sufficient, for the following reason. The perturbative formulae for the interacting fields at k-th order involve an integration over k "interaction points" in the domain of the interaction (which might be the entire spacetime). It turns out that we can control the contributions from these integrations in the OPE if we split the interaction domain into a region "close" to the points $x_1, \ldots, x_n$, and a region "far away." However, the points

$x_1, \ldots, x_n$ themselves are supposed to be scaled, i.e., they move, so the split of the integration domain has to be constantly adapted. We achieve this by dividing the interaction domain into slices of thickness 2^{-j} centered about y, where $j = 1, 2, 3, \ldots$. We find that the contributions from these slices can be controlled individually, and then be summed, if the interaction is renormalizable.

Finally, we will derive the following important general properties about the OPE coefficients C. (a) They have a local and covariant dependence upon the spacetime metric, (b) they satisfy an associativity property, (c) they satsify a renormalization group equation, (d) they satisfy a certain microlocal condition on their so-called wave front set, and (e) they can be expanded in a "scaling expansion". Let us explain these properties.

The local covariance property (a) of the OPE coefficients states that if (M, g) and (M', g') are globally hyperbolic spacetimes with corresponding OPE coefficients C resp. C', and if $f : M \to M'$ is a causality and orientation preserving isometric embedding, then f^*C' is equivalent to C at short distances. Thus, in this sense, the OPE coefficients are local functionals of the metric (and the coupling constants), and in particular do not depend upon the large scale structure of spacetime, such as the topology of M. This property can be understood from the fact that, as shown in [23, 4], the interacting fields may be constructed in a local and covariant fashion. More precisely, whenever we have a causality preserving isometric embedding f, there exists a linear map α_f from the quantum field algebra associated with (M, g) into the algebra associated with (M', g') preserving the algebraic relations. Furthermore, the fields are local and covariant in the sense [21, 5] that the image of an interacting field on (M, g) via α_f corresponds precisely to the definition of that field on (M', g'). Since the OPE may be interpreted as an asymptotic algebraic relation, and since the local and covariance property as just stated means that algebraic relations only depend on the metric locally and transform covariantly under spacetime embedding, it is natural to expect that also the OPE coefficients depend locally and covariantly upon the metric (and the couplings). This is indeed what we shall prove.

The associativity property (b) arises when one studies the different ways in which a configuration of n points can approach the diagonal in M^n. For example, in the case $n = 3$, we may consider a situation in which all three points approach each other at the same rate, or we may alternatively consider a situation in which two points approach each other faster than the third one. The possible ways in which n points can approach each other may be described by corresponding merger trees $\mathcal{T}$ which characterize the subsequent mergers [18, 1]. The associativity property states that scaling together the points in an operator product according to a given tree is equivalent to performing subsequently the OPE in the hierarchical order represented by the tree $\mathcal{T}$. We will argue—relying mainly on a general theorem of [25]—that this type of "short distance factorization property" indeed holds in perturbation theory. The associativity may again be understood intuitively from the fact that the OPE coefficients are in some sense the structure "constants" of the abstract associative *-algebra of which the interacting fields are elements.

The renormalization property (c) of the OPE coefficients arises from the fact that the

algebras of quantum fields satisfy a similar property [23]. Namely, if we rescale the metric by a constant conformal factor λ^2, then this is equivalent (in the sense of giving rise to isomorphic algebras) to redefining the field generators in a particular way, and at the same time letting the coupling constants of the interaction (which enter the structure of the algebra) flow in a particular way dictated by the "renormalization group flow" of the theory [23]. Again, since the OPE coefficients are in a sense the structure constants of the algebra, they can be expected to have a corresponding property.

The microlocal property, (d), is a property describing the nature of the singularities in an OPE coefficient C. It states that the wave front set [27] WF(C) has a characteristic form that encodes the positivity of energy momentum in the tangent space. Our condition found for WF(C) is similar in nature to the so-called "microlocal spectrum condition" [3, 35] for the wave front set of correlation functions of linear field theories in curved space. However, our condition differs qualitatively from that proposed in [3, 35] in that the interactions may affect the form of the wave front set in our case.

The scaling expansion (e) states that, if we scale n points $x_1, x_2, \ldots, x_n$ in M together according to a merger tree, then a given OPE coefficient C can be approximated to any desired precision by a finite sum of terms each of which has the form of a polynomial in the mass and curvature tensors at point $y = x_n$, times a Lorentz invariant Minkowski distribution in the Riemannian normal coordinates of $x_1, \ldots, x_{n-1}$ relative to $y = x_n$. These Minkowski distributions have, to each order in perturbation theory, a simple homogeneous scaling behavior modified by polynomials in the logarithm. They may be extracted from the given OPE coefficients by taking a certain "Mellin moment", which is an operation defined by first taking the Mellin-transform of a function and then extracting certain residue.

The general properties just described (except (e)) are postulated axiomatically in the fourthcoming paper [25], and so our present analysis may be viewed as a confirmation of these axioms in perturbation theory.

Our plan for this paper is as follows. In section 2, we first recall the general strategy for obtaining the perturbation series for the interacting fields [21, 22, 4]. The OPE is derived in section 3, and its general properties are derived in section 4. An example illustrating our algorithm for computing the OPE coefficients is presented in section 5.

As indicated, for simplicity and concreteness, we only consider a single hermitian, scalar field with renormalizable interaction (in 4 spacetime dimensions). While the restriction to a renormalizable interaction seems to be essential, we expect that our algorithm will work for other types of fields with higher spin and renormalizable interaction, in other dimensions. However, the analysis of the OPE in the physically interesting case of Yang-Mills theories would first require an understanding of the renormalization of such theories in curved spacetime, which is considerably complicated by the issue of gauge invariance. This is at present an open problem.

2 Perturbation theory

A single hermitian scalar field ϕ in 4 dimensions is described classically by the action

$$S = \int_M \left[g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + (m^2 + \alpha R)\phi^2 + 2 \sum_i \kappa_i \mathcal{O}_i \right] d\mu \,. \tag{2}$$

Here $d\mu = \sqrt{g}\, dx^0 \wedge \cdots \wedge dx^3$, $m^2, \alpha \in \mathbb{R}$, the quantities $\kappa_i \in \mathbb{R}$ are coupling parameters parametrizing the strength of the self-interaction of the field, and throughout this paper $\mathcal{O}_i$ are polynomials in the field ϕ and its covariant derivatives, as well as possibly the Riemann tensor and its derivatives. In the above action, they encode the nature of the self-interaction. Later, we will assume that the interaction is renormalizable, but for the moment no such assumption need to be made.

The perturabative construction of the quantum field theory associated with this action has been performed in a series of papers [21, 22, 24, 4]. These constructions consist of the following steps. First, one defines an abstract *-algebra [12, 21] $\mathcal{F}(M, g)$ containing the quantized field ϕ, together with its Wick powers $\mathcal{O}_i$, for the corresponding linear theory, which classically corresponds to dropping the self-interaction term

$$I = \int_M \sum_i \kappa_i \mathcal{O}_i \, d\mu = \int_M \mathcal{L} \, d\mu \,, \tag{3}$$

from the above action S. From these quantities, one then constructs the corresponding interacting quantum fields (smeared with a compactly supported testfunction) as formal power series in free field quantities via the Bogoliubov formula [19, 29],

$$\mathcal{O}_i(h)_I = \sum_{n \geq 0} \frac{\mathrm{i}^n}{n!} \mathbf{R}_n \Big(\textstyle\int_M \mathcal{O}_i h \, d\mu, \, I^{\otimes n} \Big) \,, \quad h \in C_0^\infty(M) \,, \tag{4}$$

or more formally without smearing,

$$\mathcal{O}_i(x)_I = \sum_{n \geq 0} \frac{\mathrm{i}^n}{n!} \mathbf{R}_n (\mathcal{O}_i(x), I^{\otimes n}) \,, \tag{5}$$

where the quantities $\mathbf{R}_n$ are the so-called retarded products, which are multi-linear maps on the space of local classical action functionals of the form (3), taking values in the underlying free field algebra $\mathcal{F}(M, g)$. In order to describe these constructions in more detail, it is first necessary to recall at some length the definition and key features of the linear field algebra $\mathcal{F}(M, g)$ and its quantum states, as well as nature of the retarded products $\mathbf{R}_n$.

The definition of $\mathcal{F}(M, g)$ can be stated in different equivalent ways [23, 12, 13, 14], and we now give a definition that is most suited to our purpose. Let $\omega_2 \in \mathcal{D}'(M \times M)$ be a

bidistribution of Hadamard type, meaning (a) that ω_2 is a bisolution to the Klein-Gordon equation $\Box - m^2 - \alpha R$ in each entry, (b) that the anti-symmetric part of ω_2 is given by

$$\omega_2(x_1, x_2) - \omega_2(x_2, x_1) = \mathrm{i}(\Delta_A(x_1, x_2) - \Delta_R(x_1, x_2)) \equiv \mathrm{i}\Delta(x_1, x_2)\,, \tag{6}$$

where $\Delta_{A/R}$ are the unique advanced and retarded propagators of the Klein-Gordon equation [17], and (c) that it has a wave front set [27] of Hadamard type [35]:

$$\mathrm{WF}(\omega_2) = \{(x_1, k_1; x_2, k_2) \in T^*(M \times M) \setminus 0;\ k_1 = p, k_2 = -p, p \in \bar{V}_+^*\}\,. \tag{7}$$

Here, it is understood that the set can only contain those x_1 and x_2 that can be joined by a null geodesic, γ, and that $p = p_\mu dx^\mu$ is a parallel co-vector field tangent to that null geodesic, meaning that $\nabla_{\dot\gamma} p = 0$, and $V_\pm^*$ is the dual of the future/past lightcone. The desired Wick-polynomial algebra $\mathcal{F}(M, g)$ is now generated by an identity $\mathbb{1}$, and the following symbols F:

$$F = \int_{M^n} \left(:\phi^{\otimes n}:_\omega\ u_n\right)(x_1, \ldots, x_n) \bigwedge_{i=1}^n d\mu(x_i)\,, \tag{8}$$

where u_n is a symmetric, compactly supported distribution on M^n subject to the wave front condition [12]

$$\mathrm{WF}(u_n) \cap [(\bar{V}_+^*)^n \cup (\bar{V}_-^*)^n] = \emptyset\,. \tag{9}$$

The relations in the *-algebra $\mathcal{F}(M, g)$ are as follows: The *-operation is defined by letting F^* be given by the same expression as F, but with u_n replaced by its complex conjugate, and the product is defined by

$$\begin{aligned} &:\phi^{\otimes n}:_\omega (x_1, \ldots, x_n) : \phi^{\otimes m}:_\omega (x_{n+1}, \ldots, x_{n+m}) = \\ &\sum_k \frac{n!m!}{(n-k)!(m-k)!k!} \sum_{p_1, \ldots, p_k \in P} \prod_i \omega_2(x_{p_i(1)}, x_{p_i(2)}) : \phi^{\otimes n+m-2k}:_\omega (\{x_j;\ j \notin |P|\}) \end{aligned} \tag{10}$$

where P is the set of all pairs $p_i \in \{1, \ldots, n\} \times \{n+1, \ldots, n+m\}$. This formula is identical in nature to the standard Wick theorem for normal ordered quantities (relative to a Gaussian state with 2-point function ω_2). The wave front conditions on u_n and ω_2 are needed in order guarantee that the product between the corresponding integrated quantities as in (8) exists, because the latter involves the pointwise products of distributions [27]. The definition of the algebra $\mathcal{F}(M, g)$ superficially seems to depend on the particular choice of ω_2, but this is in fact not so: A change of ω_2 merely corresponds to a relabeling of the generators, and does not change the definition of $\mathcal{F}(M, g)$ as an abstract algebra [21].

The relation between the abstract quantization of the linear Klein-Gordon field just described and more familiar ones is as follows. If f is a smooth test function, then the generator (8) with $n = 1$ and $u_1 = f$ can be identified with the smeared field $\phi(f)$. Indeed,

using the Wick formula (10), and the antisymmetric part of ω_2, eq. (6), one easily derives the relation $[\phi(f_1), \phi(f_2)] = \mathrm{i}\Delta(f_1, f_2)\mathbb{1}$, which is the standard commutation relation for a linear scalar field in curved spacetime. The higher order generators (8) with u_n given by the n-fold tensor product $f^{\otimes n} = f \otimes \cdots \otimes f$ of a smooth testfunction correspond to a smeared normal ordered product $:\phi^{\otimes n}:_\omega (f^{\otimes n})$, formally related to the field ϕ itself by

$$:\phi^{\otimes n}:_\omega (x_1, \ldots, x_n) = \frac{\delta^n}{\mathrm{i}^n \delta f(x_1) \cdots \delta f(x_n)} \exp\left\{\mathrm{i}\phi(f) + \frac{1}{2}\omega_2(f,f)\right\}\Big|_{f=0} . \qquad (11)$$

As it stands, the smeared version of the Klein-Gordon equation $\phi((\Box - m^2 - \alpha R)f) = 0$ is not an algebraic relation. However, this relation could easily be incorporated by factoring $\mathcal{F}(M,g)$ by the 2-sided *-ideal $\mathcal{J}(M,g)$ consisting of all elements F of the form (8) with u_n a distribution in the class (9) which is in the image of this distribution class under the Klein-Gordon operator, such as $(\Box - m^2 - \alpha R)f$ in the simplest case. The purpose of this paper will be to establish an OPE, and it is technically convenient for this purpose not to factor by the ideal. However, after the OPE has been constructed there is absolutely no problem to factor by this ideal, because it is clear that the OPE will continue to hold on the factor algebra.

Quantum states in the algebraic framework are linear expectation functionals $\Phi : \mathcal{F}(M,g) \to \mathbb{C}$ that are normalized, meaning $\Phi(\mathbb{1}) = 1$, and of positive type, meaning $\Phi(F^*F) \geq 0$. Of particular importance are the so-called Hadamard states on $\mathcal{F}(M,g)$. Those states are defined by the fact that their 2-point function $\Phi_2(x_1, x_2) = \Phi(\phi(x_1)\phi(x_2))$ satisfies properties (a), (b), and (c) listed above, and that their truncated n-point functions of the field ϕ for $n \neq 2$ are smooth solutions to the Klein-Gordon equation. The key consequence of the Hadamard requirement which we shall need later is [20] that $\Phi(:\phi^{\otimes n}:_\omega (x_1, \ldots, x_n))$ is smooth. Note that by definition, the n-point functions of a Hadamard state satisfy the Klein-Gordon equation. Consequently, they vanish on the ideal $\mathcal{J}(M,g)$ generated by the Klein-Gordon equation and hence induce states on the factor algebra.

Later, we want to define an operator product expansion, and for this we will need a notion of what it means for algebra elements to be "close" to each other. For this we now introduce a topology on $\mathcal{F}(M,g)$. There are various ways to do that. A particular topology was introduced in [21]. We prefer here to work with a different (weaker) topology, defined by the collection of all linear functionals on $\mathcal{F}(M,g)$ with the property that $\Phi(:\phi^{\otimes n}:_\omega (x_1, \ldots, x_n))$ is smooth. This set includes the Hadamard states as defined above, and we shall, by abuse of notation, sometimes refer to such Φ as "Hadamard" as well. We then introduce a set of seminorms $\mathcal{N}_\Phi(F) = |\Phi(F)|$, labelled by these functionals Φ. We say that a sequence $\{F_N\}_{N\in\mathbb{N}}$ of algebra elements tends to zero if for each Φ, and each $\varepsilon > 0$, there is an N_0 such that $\mathcal{N}_\Phi(F_N) \leq \varepsilon$ for all $N \geq N_0$.

An important feature of the algebra $\mathcal{F}(M,g)$ is that it has a local and covariant dependence upon the spacetime. More precisely, if $f : M \to M'$ is a causality and orientation preserving isometric embedding of a globally hyperbolic spacetime (M,g) into

another such spacetime (M', g'), then there exists a continuous, injective *-homomorphism

$$\alpha_f : \mathcal{F}(M, g) \longrightarrow \mathcal{F}(M', g') \,. \tag{12}$$

This embedding is most simply described in terms of its action on a smeared field $\phi(h)$, where h is a test function on M. If $h' = f_* h$ is the corresponding pushed forward test function on M', then we define $\alpha_f[\phi(h)] = \phi(h')$. Furthermore, the action of α_f on an arbitrary element in $\mathcal{F}(M, g)$ may then be defined by continuity, because the subalgebra generated by expressions of the form $\phi(h)$ is dense in $\mathcal{F}(M, g)$. The action of α_f on the smeared field $\phi(h)$ is characteristic for so-called "local covariant fields". Namely, an algebra valued distribution $\mathcal{O}_i : C_0^\infty(M) \to \mathcal{F}(M, g)$, $h \mapsto \mathcal{O}_i(h)$ that is defined for *all* spacetimes (M, g) is called a (scalar) local and covariant field if

$$\alpha_f[\mathcal{O}_i(h)] = \mathcal{O}_i(h') \quad h' = f_* h \,, \tag{13}$$

whenever f is an orientation and causality preserving isometric embedding. Local covariant fields of tensor type are defined in the same way, except that the testfunction h is now a section in the (dual of the) vector bundle $\mathbf{E}_i$ corresponding to the tensor type. Thus, the field ϕ is (by definition) a local and covariant field. On the other hand, the normal ordered n-th Wick power of a field defined by putting $u_n(x_1, \ldots, x_n) = f(x_n)\delta(x_1, \ldots, x_n)$ in eq. (8) is not a local and covariant field, because it implicitly depends on the choice of the 2-point function ω_2, which is not a local and covariant quantity [21]. The possible definitions of Wick powers giving rise to local and covariant fields (satisfying also various other natural conditions) were classified in [21]. It turns out that the definition of a given classical expression

$$\mathcal{O}_i = \nabla^{a_1}\phi\nabla^{a_2}\phi\cdots\nabla^{a_n}\phi, \quad i = \{a_1, a_2, \ldots, a_n\} \in \mathcal{I} \equiv \bigoplus_n \mathbb{Z}^n_{\geq 0} \tag{14}$$

as a local, covariant field in $\mathcal{F}(M, g)$ is not unique, but contains certain ambiguities. As proven in [21], these ambiguities correspond to the possibility of adding to a given field lower order Wick power powers times certain polynomials of the Riemann tensor and its derivatives $\nabla_{(\mu_1}\cdots\nabla_{\mu_k)}R_{\nu_1\nu_2\nu_3\nu_4}$ of the same dimension as $\mathcal{O}_i$. Here, the dimension of a Wick power is the map $[\cdot] : \mathcal{I} \to \mathbb{N}$ from the index set labelling the various fields, into the natural numbers defined by

$$[i] = n + \sum_{i=1}^{n} a_i \,. \tag{15}$$

One definition which is local and covariant (and satisfies also the other natural conditions given in [21]) is the following "local normal ordering prescription". It is based upon the use of the local Hadamard parametrix H, which is the bidistribution defined on a convex

normal neighborhood of the diagonal $\{(x,x);\ x\in M\}$ of $M\times M$ by[2]

$$H=\frac{v_0}{\sigma+\mathrm{i}0t}+\left(\sum_{n\geq 0}\frac{1}{2^n n!}v_{n+1}\sigma^n\right)\ln(\sigma+\mathrm{i}0t)\,. \tag{16}$$

In this expression, $\sigma(x,y)$ is the signed squared geodesic distance between two points, we have defined $t(x,y)=\tau(x)-\tau(y)$, where τ is a time-function, and the $v_n(x,y)$ are smooth symmetric functions that are determined by requiring that H be a parametrix, i.e., a solution to the Klein-Gordon equation $\Box-m^2-\alpha R$ in each entry modulo a smooth remainder. Explicitly $v_0=D^{1/2}/2\pi^2$ is given in terms of the VanVleck determinant D, defined by

$$D=-\frac{1}{4}\frac{|(\nabla\otimes\nabla)\sigma|}{|\mathcal{J}|}\,. \tag{17}$$

Here, $\mathcal{J}(x,y)\in T^*_xM\otimes T^*_yM$ is the bitensor of parallel transport, $(\nabla\otimes\nabla)\sigma(x,y)\in T^*_xM\otimes T^*_yM$ is the is the bitensor obtained by taking the gradient of $\sigma(x,y)$ in both x and y, and we are defining a biscalar $|\mathcal{B}(x,y)|$ by

$$|\mathcal{B}(x,y)|\,d\mu(x)\otimes d\mu(y)=\bigwedge^4\mathcal{B}(x,y) \tag{18}$$

from any bitensor $\mathcal{B}(x,y)$. The smooth functions $v_n(x,y)$ are iteratively defined by the transport equations [10]

$$2(\nabla^\mu\sigma)\nabla_\mu v_n+[(\nabla^\mu\sigma)\nabla_\mu\ln D+4n]v_n=-2(\Box-m^2-\alpha R)v_{n-1}\,, \tag{19}$$

where the derivatives act on x. These functions are symmetric in x and y [34, 17], and their germs at the diagonal are locally and covariantly defined in terms of the metric. Where it is well-defined, H has a wave front set of Hadamard type (7). Next, fix a convex normal neighborhood of the diagonal in M^n, and in that neighborhood define locally normal ordered products $:\phi^{\otimes n}:_{\mathrm{H}}(x_1,\ldots,x_n)$ by the same formula as (11), but with ω_2 in that formula replaced by H,

$$:\phi^{\otimes n}:_{\mathrm{H}}(x_1,\ldots,x_n)=\frac{\delta^n}{\mathrm{i}^n\delta f(x_1)\cdots\delta f(x_n)}\exp\left\{\mathrm{i}\phi(f)+\frac{1}{2}H(f,f)\right\}\Big|_{f=0}\,. \tag{20}$$

As the expressions (11), their expectation value in any Hadamard state is smooth. Following [21] we define the local covariant n-th Wick power of the field as the distribution valued in $\mathcal{F}(M,g)$ given by

$$\phi^n(x)=\lim_{\varepsilon\to 0}:\phi^{\otimes n}:_{\mathrm{H}}(\vec{x}(\varepsilon))\quad\vec{x}(\varepsilon)=(\exp_x(\varepsilon\xi_1),\ldots,\exp_x(\varepsilon\xi_n))\,, \tag{21}$$

[2]The infinite sum is to be understood in the sense of an asymptotic expansion.

where ξ_i denote the Riemannian normal coordinates (identified with a vector in $\mathbb{R}^4$) of the point x_i relative to x. More generally, fields containing derivatives are defined by first acting with the derivatives on the appropriate tensor factor in $:\phi^{\otimes n}:_{\mathrm{H}}$ before taking the above "coincidence limit". That is, if $i = (a_1, \ldots, a_n) \in \mathcal{I}$ denotes a collection of natural numbers, then the corresponding local covariant field $\mathcal{O}_i(x)$ is defined by applying the partial derivative operator $\partial_{\xi_1}^{a_1} \cdots \partial_{\xi_n}^{a_n}$ prior to the coincidence limit. Note that this definition is covariant, because partial derivatives with respect to Riemannian normal coordinates at x may be expressed in terms of curvature tensors at x and covariant derivatives ∇.

Having described the algebra $\mathcal{F}(M, g)$ of field observables in the linear field theory associated with the action S without the interaction terms, we now turn to the interacting theory. For this it is technically convenient at an intermediate step to assume that the couplings κ_i in the action S are not constants, but actually smooth functions of compact support in M, which we assume are locally constant,

$$\kappa_i(x) = \kappa_i \chi(x)\,, \tag{22}$$

where $\chi \in C_0^\infty(M)$, and $\chi(x) = 1$ in an open set in M with compact closure. The cutoff functions χ serve as an infra red cutoff and are removed at a later stage. With the introduction of cutoff functions understood, the interacting fields are defined by the Bogoliubov formula (5) in terms of the retarded products $\mathbf{R}_n$. Each retarded product is a continuous, bilinear map

$$\mathbf{R}_n : \mathcal{S} \times \left(\bigotimes^n \mathcal{S} \right) \to \mathcal{F}(M, g) \quad (I_0, I_1 \otimes \cdots \otimes I_n) \mapsto \mathbf{R}_n(I_0, I_1 \otimes \cdots \otimes I_n)\,, \tag{23}$$

from the tensor powers of the space $\mathcal{S}$ of all classical action functionals $I_j = \int \mathcal{L}_j \, d\mu$ that are local and polynomial in the field ϕ and whose couplings are compactly supported functions on M. The map is taking values in the algebra $\mathcal{F}(M, g)$ associated with the linear field theory, and is symmetric in $I_1, \ldots, I_n$. Note that the power series expression (5) for the interacting fields is only a formal series, and no statement is made about its convergence. Since each term in these series is an element in $\mathcal{F}(M, g)$, the interacting fields are elements of the algebra $\mathcal{P} \otimes \mathcal{F}(M, g)$, where

$$\mathcal{P} \equiv \mathbb{C}[\![\kappa_1, \kappa_2, \ldots]\!] = \left\{ \sum_{\alpha_i \geq 0} a_{\alpha_1 \ldots \alpha_k} \kappa_1^{\alpha_1} \cdots \kappa_k^{\alpha_k}; \quad a_{\alpha_1 \ldots \alpha_k} \in \mathbb{C} \right\} \tag{24}$$

is the corresponding ring of formal series. All operations, such as multiplication in this ring and in the algebra $\mathcal{P} \otimes \mathcal{F}(M, g)$, are defined by simply formally multiplying out the corresponding formal series term by term. Furthermore, this algebra inherits a natural topology from $\mathcal{F}(M, g)$: A formal power series converges to another formal power series in the algebra if each coefficient does.

We want the zeroth order contribution $\mathbf{R}_0(\mathcal{O}_i(x))$ in the Bogoluibov formula (5) to be given by our definition of Wick powers $\mathcal{O}_i(x)$ in the linear field theory[3]. Thus, we define $\mathbf{R}_0(\mathcal{O}_i(x))$ to be equal to the locally normal ordered field $\mathcal{O}_i(x)$ given in eq. (21). The terms with $n \geq 1$ in the formal series (5) represent the perturbative corrections coming from the interaction, I. They involve the higher, non-trivial, retarded products. The construction of these retarded products can be reduced[4] to the construction of the so-called "time-ordered products," because there exists a well-known formula of combinatorial nature relating these two quantities, see e.g. the appendix of [14]. The construction of the time-ordered products in turn has been given in [21, 22, 24], which is based on work of [4]. The strategy in these papers is to first write down a number of *functional relations* for the time ordered products that are motivated by corresponding properties of the interacting fields defined by the Bogoliubov formula. These properties then dictate to a large extent the construction of the time-ordered (and hence the retarded) products. Since there is a combinatorial formula relating the time ordered products to the retarded products, these relations can be equivalently be stated in terms of the retarded products. The relevant relations[5] for this paper are as follows:

(r1) **Causality:** Let $F, G, S_i \in \mathcal{S}$. Suppose that there is a Cauchy surface such that $\operatorname{supp} G$ is in its future and $\operatorname{supp} F$ in its past. Then

$$\mathbf{R}_n\left(F, G \otimes \bigotimes_i S_i\right) = 0. \tag{25}$$

(r2) **GLZ factorization formula** [14]:

$$\begin{aligned}\mathbf{R}_n\left(G, \bigotimes_i S_i \otimes F\right) - \mathbf{R}_n\left(F, \bigotimes_i S_i \otimes G\right) \\ = \sum_{I\cup J=\{1,\ldots,n-1\}} \left[\mathbf{R}_{|I|,1}\left(F, \bigotimes_{i\in I} S_i\right), \mathbf{R}_{|J|,1}\left(G, \bigotimes_{j\in J} S_j\right)\right].\end{aligned} \tag{26}$$

(r3) **Expansion:** There exist local covariant c-number distributions r near the total

[3]Note that the argument of the retarded product $\mathbf{R}_0(\mathcal{O}_i(x))$ is a *classical* action (or density), while the corresponding Wick power $\mathcal{O}_i(x)$ is a distribution valued in the quantum algebra $\mathcal{F}(M, g)$. We should strictly speaking distinguish these quantities by introducing a new notation for the Wick power, but we shall not do this for simplicity.

[4]Alternatively, it should also be possible to construct the retarded products directly along the lines of [12, 13, 14], by suitably generalizing the arguments of that paper from Minkowski spacetime to curved spacetime.

[5]A complete list may be found in [24].

diagonal $\{(x, x, \ldots, x);\ x \in M\}$ such that (with $l_j < i_j$)

$$\mathbf{R}_n(\mathcal{O}_{i_0}(x), \mathcal{O}_{i_1}(y_1) \cdots \mathcal{O}_{i_n}(y_n)) = \sum_{l_0, l_1, \ldots, l_n} r^{l_0 l_1 \ldots l_n}_{i_0 i_1 \ldots i_n}(x, y_1, \ldots, y_n) \ : \mathcal{O}_{l_0}(x) \mathcal{O}_{l_1}(y_1) \ldots \mathcal{O}_{l_n}(y_n) :_{\mathrm{H}} \ . \quad (27)$$

(r4) **Scaling degree:** The distributions r have the scaling degree

$$\mathrm{sd}(r^{l_0 l_1 \ldots l_n}_{i_0 i_1 \ldots i_n}) = \sum_k [i_k] - [l_k] \quad (28)$$

at the total diagonal $\{(x, x, \ldots, x);\ x \in M\} \subset M^{n+1}$. Here, the scaling degree of a distribution $u \in \mathcal{D}'(X)$ at a submanifold $Y \subset X$ is defined as follows [37, 4]. Let $S_\varepsilon : X_0 \to X_0$ be an injective, smooth map defined on an open neighborhood X_0 of Y with the properties (a) that $S_\varepsilon \restriction_Y = id_Y$, and (b) that for all $y \in Y$, the map $(\mathrm{D}S_\varepsilon)(y) : T_yX \to T_yX$ is the identity on T_yY and scales vectors by $\varepsilon > 0$ on a complementary subspace $C_y \subset T_yX$ of T_yY. Then u has scaling degree $\mathrm{sd}(u)$ at Y if $\lim_{\varepsilon \to 0} \varepsilon^D u \circ S_\varepsilon = 0$ for all $D > \mathrm{sd}(u)$ in the sense of $\mathcal{D}'(X_0)$. The definition is independent of the precise choice of S_ε.

(r5) **Locality and covariance:** Let $f : (M, g) \to (M', g')$ be a causality and orientation preserving isometric embedding. Then the retarded products satisfy

$$\alpha_f \left[\mathbf{R}_n \left(F, \bigotimes_i S_i \right) \right] = \mathbf{R}_n \left(f_* F, \bigotimes_i f_* S_i \right) , \quad (29)$$

where f_* denotes the natural push-forward of a local action functional on M to the corresponding action functional on M'.

(r6) **Microlocal condition:** The distributions r in the expansion (27) have the following wave front set:

$$\begin{aligned} \mathrm{WF}(r) \ \subset \ \Big\{ & (x, k; y_1, l_1; \ldots; y_m, l_m) \in T^*M^{m+1}; \text{there is graph in } \mathcal{G}_{1,n} \text{ such that} \\ & k = \sum_{e : s(e) = x} p_e - \sum_{e : t(e) = x} p_e, \quad l_i = \sum_{e : s(e) = y_i} p_e - \sum_{e : t(e) = y_i} p_e \\ & y_i \in J^-(x) \quad i = 1, \ldots, m \Big\} . \end{aligned} \quad (30)$$

The valence of the vertices y_i in the graph is restricted to be less or equal than the maximum power of ϕ occurring an the operators $\mathcal{O}_{i_1}, \ldots, \mathcal{O}_{i_n}$ in eq. (27).

In the formulation of the last condition, we are using a graph theoretical notation [3], which will be useful later as well. Most generally, we consider the set $\mathcal{G}_{n,m}$ of embedded, oriented graphs in the spacetime M with $n+m$ vertices. Each such graph has n so-called "external vertices", $x_1, \ldots, x_n \in M$, and m so-called "internal" or "interaction vertices" $y_1, \ldots, y_m \in M$. These vertices are of arbitrary valence, and are joined by edges, e, which are null-geodesic curves $\gamma_e : (0,1) \to M$. It is assumed that an abstract ordering[6] $<$ of the vertices is defined, and that the ordering among the external vertices is $x_1 < \cdots < x_n$, while the ordering of the remaining interaction vertices is unconstrained. If e is an edge joining two vertices, then $s(e)$ (the source) and $t(e)$ (the target) are the two vertices $\gamma_e(0)$ and $\gamma_e(1)$, where the curve is oriented in such a way that it starts at the smaller vertex relative to the fixed vertex ordering. Each edge carries a future directed, tangent parallel covector field, p_e, meaning that $\nabla_{\dot\gamma_e} p_e = 0$, and $p_e \in \partial V_+^*$.

Similar to the case of local covariant Wick products (the case $n = 0$), the above functional relations (together with other functional relations described in detail in [21, 22]) do not uniquely fix the retarded products: There remains a number of real constants at each order n which parametrize the set of possible definitions of $\mathbf{R}_n$ that are compatible with (r1)–(r6). These correspond to the usual "renormalization ambiguities" in perturbative quantum field theory, see [22, 23, 24] for details.

One finally needs to remove the dependence of the interacting fields on the arbitrary cutoff function χ. For this, one investigates how the interacting field changes when the cutoff function is varied. Assume that χ_1 and χ_2 are two different cutoff functions, both of which are equal to 1 in an open globally hyperbolic neighborhood $U \hookrightarrow M$. Let I_1 and I_2 be the corresponding interactions. Note that, as a classical functional, the difference $I_1 - I_2$ is supported in a compact region, and vanishes in the neighborhood U where the cutoff functions coincide. The key fact [4], which follows from the above functional relations (r1) and (r2), is now that there exists a unitary operator $V \in \mathcal{P} \otimes \mathcal{F}(M,g)$, depending upon I_1, I_2, with the property that

$$\mathcal{O}_i(x)_{I_1} = V\, \mathcal{O}_i(x)_{I_2}\, V^*, \quad \text{for all } x \in U,\, i \in \mathcal{I}. \tag{31}$$

This relation may be interpreted as saying that the algebraic relations between the interacting fields within the region U where the cutoff function is constant do not depend on how the cutoff function is chosen outside this region, and this observation may be used to construct an abstract interacting field algebra associated with the entire spacetime (M,g) that is independent of the choice of cutoff function [4]. However, in the present context, we are actually only interested in a small patch U of spacetime where we want to consider the OPE. Therefore, it will be more convenient for us to simply fix an arbitrary cutoff function that is equal to 1 in the patch U of interest. Since the OPE concerns only local algebraic relations, it is clear from (31) that it should not matter what cutoff function we choose, and this will formally be shown below in item 2) of section 4.

[6]The ordering is not assumed to be related to the causal structure of the manifold at this stage.

3 Operator product expansion

We will now show that the interacting fields $\mathcal{O}_i(x)_I$ described in the previous section obey an operator product expansion,

$$\mathcal{O}_{i_1}(x_1)_I \mathcal{O}_{i_2}(x_2)_I \cdots \mathcal{O}_{i_n}(x_n)_I \sim \sum_{[k]\leq\Delta} C_{i_1 i_2 \ldots i_n}{}^k(x_1, x_2, \ldots, x_n)_I \, \mathcal{O}_k(x_n)_I \,, \tag{32}$$

where C_I are certain distributinal coefficients depending upon the interaction, I, which are to be determined, and where $[k]$ is the standard dimension function defined above. We mean by the above expression that, as the points $x_1, x_2, \ldots, x_n$ approach each other, the algebra product of the interacting fields on the left side can be approximated, to the desired precision determined by Δ, by the right side, in the topology on the algebra $\mathcal{P} \otimes \mathcal{F}(M, g)$.

To make this statement precise, we must, however, take into account that both sides of the OPE are actually distributional, and that a configuration $\vec{x} = (x_1, \ldots, x_n) \in M^n$ of n mutually distinct points on a manifold may "merge" in qualitatively different manners when $n > 2$, because the points may approach each other at "different rates". The appropriate mathematical framework to formalize in a precise manner the possibility of configuration of points to approach each other at different rates is provided by a construction referred to as the "compactification of configuration space," due to Fulton and MacPherson [18], and Axelrod and Singer [1]. Let

$$\begin{aligned} M_0^n &= \{\vec{x} = (x_1, x_2, \ldots, x_n) \in M^n; x_i \neq x_j\} && (33) \\ &= \{\vec{x} \in \mathrm{Map}(\{1, \ldots, n\}, M), \quad \vec{x}(i) = x_i; \quad \vec{x} \text{ injective}\} && (34) \end{aligned}$$

be the configuration space, i.e., the space of all configurations of n mutually distinct points in M. The union of partial diagonals

$$\partial M_0^n = \bigcup_{S \subset \{1,\ldots,n\}} \Delta_S \subset M^n \tag{35}$$

where a partial diagonal is defined by

$$\Delta_S = \{\vec{x} \in \mathrm{Map}(\{1, \ldots, n\}, M); \ \vec{x} \restriction S = \text{constant}\} \tag{36}$$

is the boundary of the configuration space M_0^n. Configurations of points where some points come close to each other are in some sense close to this boundary. The Fulton-MacPherson compactification $M[n]$ is obtained by attaching a different boundary, $\partial M[n]$, to M_0^n, which in addition incorporates the various directions in which ∂M_0^n can be approached. This boundary may be characterized as the collection of endpoints of certain curves $\vec{x}(\varepsilon)$, in $M[n]$, which are in M_0^n for $\varepsilon > 0$, and which end on $\partial M[n]$ at $\varepsilon = 0$. These curves are labeled by trees $\mathcal{T}$ that characterize subsequent mergings of the points in the configuration

as $\varepsilon \to 0$. A convenient way to describe a tree $\mathcal{T}$ (or more generally, the disjoint union of trees, a "forest") is by a nested set $\mathcal{T} = \{S_1, \ldots, S_k\}$ of subsets $S_i \subset \{1, \ldots, n\}$. "Nested" means that two sets are either disjoint, or one is a proper subset of the other. We agree that the sets $\{1\}, \ldots, \{n\}$ are always contained in the tree (or forest). Each set S_i in $\mathcal{T}$ represents a node of a tree, i.e., the set of vertices $\mathrm{Vert}(\mathcal{T})$ is given by the sets in $\mathcal{T}$, and $S_i \subset S_j$ means that the node corresponding to S_i can be reached by moving downward from the node represented by S_j. The root(s) of the tree(s) correspond to the maximal elements, i.e., the sets that are not subsets of any other set. If the set $\{1, \ldots, n\} \in \mathcal{T}$, then there is in fact only one tree, while if there are several maximal elements, then there are several trees in the forest, each maximal element corresponding to the root of the respective tree. The leaves correspond to the sets $\{1\}, \ldots, \{n\}$, i.e., the minimal elements.

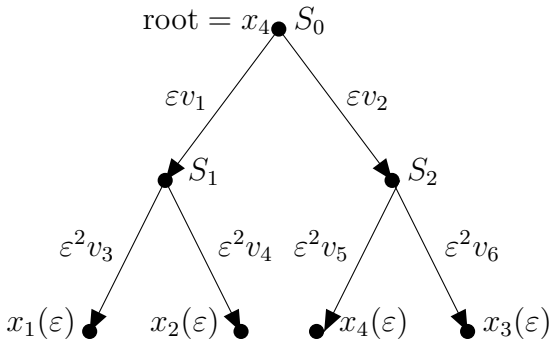
The desired curves $\vec{x}(\varepsilon)$ tending to the boundary of $M[n]$ are associated with trees and are constructed as follows. With the root(s) of the tree(s), we associate a point $x_i \in M$, where i is a label that runs through the different maximal elements, while with each edge $e \in \mathrm{Edge}(\mathcal{T})$ of a given tree (a line joining two nodes), we associate a vector $v_e \in T_{x_i} M$, where x_i is associated with the root of the tree that e belongs to. To describe the definition of this vector it is convenient to identify an edge $e \in \mathrm{Edge}(\mathcal{T})$ with the pair $e = (S, S')$ of nodes that it connects, i.e., an edge defines a relation in $\mathcal{T} \times \mathcal{T}$. If $S \subset S'$, then we write $S' = t(e)$ for target, and $S = s(e)$, for the source. We then set

$$v_e = \xi_{m(t(e))} - \xi_{m(s(e))} \quad m(S) = \max\{i;\ i \in S\}\,, \tag{37}$$

where ξ_j denotes the Riemannian normal coordinates (identified with a vector in $\mathbb{R}^4$ via a choice of orthonormal tetrad at the corresponding root x_i). We define the desired curve $\vec{x}(\varepsilon)$ by

$$x_j(\varepsilon) = \sum_{e \in p_j} v_e \varepsilon^{\mathrm{depth}(t(e))}\,. \tag{38}$$

where p_j is the unique path connecting the leaf j with the corresponding root, where v_e is given in terms of $\vec{x}$ by eq. (37), and where $depth(S)$ is the number of edges that connect the node $S \in \mathcal{T}$ with the root. The following figure illustrates this definition in an example:



$$\mathcal{T} = \{S_0, S_1, \ldots, S_6\}$$
$$x_1(\varepsilon) = \varepsilon v_1 + \varepsilon^2 v_3,\ x_2(\varepsilon) = \varepsilon v_1 + \varepsilon^2 v_4,\ x_3(\varepsilon) = \varepsilon v_2 + \varepsilon^2 v_5,\ x_4(\varepsilon) = \varepsilon v_2 + \varepsilon^2 v_6$$

For each fixed tree $\mathcal{T}$, and each fixed ε, the above curve defines a map

$$\psi_{\mathcal{T}}(\varepsilon) : M_0^n \to M_0^n, \quad \vec{x} \mapsto \vec{x}(\varepsilon) \tag{39}$$

flowing the point $\vec{x} = \vec{x}(1)$ to the point $\vec{x}(\varepsilon)$. For $\varepsilon = 0$, the image of this map may be viewed as a portion of the boundary $\partial M[n]$ corresponding to the tree. The roots of $\mathcal{T}$ correspond to the particular diagonal; in particular, if there is only one tree in $\mathcal{T}$ (as we shall assume from now on) then the configuration $\vec{x}(\varepsilon)$ converges to the total diagonal in M^n. It may be checked that the maps $\psi_{\mathcal{T}}(\varepsilon)$ satisfy the composition law

$$\psi_{\mathcal{T}}(\varepsilon) \circ \psi_{\mathcal{T}}(\varepsilon') = \psi_{\mathcal{T}}(\varepsilon\varepsilon') \,. \tag{40}$$

Using the maps $\psi_{\mathcal{T}}(\varepsilon)$ we can define an asymptotic equivalence relation $\sim_{\delta,\mathcal{T}}$ for distributions on M^n. Consider distributions u_1, u_2 defined on M^n. For a given tree $\mathcal{T}$ and $\delta > 0$, we declare the equivalence relation $\sim_{\mathcal{T},\delta}$ by

$$u_1 \sim_{\mathcal{T},\delta} u_2 \quad :\Longleftrightarrow \quad \lim_{\varepsilon \to 0+} \varepsilon^{-\delta} \, (u_1 - u_2) \circ \psi_{\mathcal{T}}(\varepsilon) = 0 \ , \tag{41}$$

in the sense of distributions on M^n, where we view $\psi_{\mathcal{T}}(\varepsilon)$ as a map $M^n \to M^n$ that is parametrized by $\varepsilon > 0$.

Having defined the equivalence relation $\sim_{\delta,\mathcal{T}}$ we can now state precisely our notion of an OPE. Namely, we require that, for each $\delta > 0$, each given set of operators, and each tree $\mathcal{T}$, there exists a Δ so that the OPE holds in the sense of $\sim_{\delta,\mathcal{T}}$. The only issue that we have not yet been quite precise about is that the OPE is not a relation between c-number distributions, but instead distributions valued in the topological algebra $\mathcal{P}\otimes\mathcal{F}(M,g)$. This difficulty is simply dealt with by requiring convergence in the equivalence relation (41) (now for algebra valued objects) with respect to the topology in the algebra. Thus, we define the precise sense in which the OPE is supposed to hold to be that for each tree $\mathcal{T}$

with one root, and each δ, there exists a $\Delta \in \mathbb{R}$ such that (32) holds in the sense of $\sim_{\delta,\mathcal{T}}$ as a relation between the corresponding algebra valued distributions.

We now come to the actual construction of the operator product coefficients in perturbation theory. As also described in [25], and as originally suggested by Bostelmann [2] and Fredenhagen and Hertel [15] in the context of algebraic quantum field theory on Minkowski spacetime, it is convenient to think of the operator product coefficients as arising via certain "standard functionals"

$$\Psi^i_{M,x}(\,.\,)_I : \mathcal{P} \otimes \mathcal{F}(M,g) \longrightarrow \mathcal{P} \otimes \mathbf{E}_i|_x \,. \tag{42}$$

These functionals depend upon the given spacetime (M,g), the index label $i \in \mathcal{I}$ describing a composite field, a point $x \in M$, and the interaction, as indicated by the subscript "I". The functionals take values in the fiber over x in the vector bundle $\mathbf{E}_i$ (viewed as a $\mathcal{P}$-module) associated with the tensor character of the field $\mathcal{O}_i$. In our constructions below, the functionals are in fact only defined on the subalgebra $\mathcal{P} \otimes \mathcal{F}(U,g)$ corresponding to a convex normal neighborhood $U \subset M$. However, since all of our considerations are entirely local, we may assume without loss of generality and to save writing that $U = M$.

The OPE coefficients are supposed to be given in terms of the above standard functionals by

$$C_{i_1 i_2 \ldots i_n}{}^j(x_1, x_2, \ldots, x_n)_I = \Psi^j_{x_n}\left(\mathcal{O}_{i_1}(x_1)_I \mathcal{O}_{i_2}(x_2)_I \cdots \mathcal{O}_{i_n}(x_n)_I\right)_I \,. \tag{43}$$

We will construct the OPE coefficients in perturbation theory by presenting a suitable set of such standard functionals. We are going to choose these standard functionals as a "dual basis" to the interacting fields, in the sense that we wish them to satisfy

$$\Psi^i_x(\mathcal{O}_j(x)_I)_I = \delta^i{}_j \,\mathrm{id}_{\mathbf{E}_i} \quad \text{for all } x \in M \text{ and } [i],[j] < \Delta. \tag{44}$$

This ansatz is motivated by the following simple consideration. Let us assume that an OPE exists. Let us fix a $\Delta > 0$, carry the OPE out until $[k] \leq \Delta$, and apply the functionals $\Psi^j_{x_n}$ to it, where $[j] \leq \Delta$. Using (44), we immediately find that the coefficients in the OPE must be given by (43), up to a remainder term coming from the remainder in the OPE. But this remainder is by assumption small for asymptotically short distances, in the sense of the above equivalence relation, provided we make Δ sufficiently large. It can therefore be ignored.

Thus we have argued that if an OPE exists in the sense above, and if standard functionals satisfying (44) have been defined, then the OPE coefficients C ought to be given by (43). Consequently, our first step will be to define the standard functionals as formal power series in the coupling constants κ_i so that eq. (44) will be satisfied to arbitrary orders in perturbation theory. To zeroth order in perturbation theory, such standard functionals are defined as follows (see also [25]). Recall that a general algebra element $F \in \mathcal{F}(M,g)$ can be written as in eq. (8) in terms of normal ordered generators (11). If

we are interested only in elements F so that the corresponding u_n in (8) are suppored sufficiently close to the diagonal in M^n (as we will always assume in the following), then we may rewrite F in terms of the locally normal ordered generators $:\phi^{\otimes n}:_{\mathrm{H}}(x_1,\ldots,x_n)$ given in eq. (20) instead of the normal ordered generators $:\phi^{\otimes n}:_{\omega}(x_1,\ldots,x_n)$. The action of the zeroth order standard functionals is then declared by

$$\Psi^i_x\left(:\phi^{\otimes m}:_{\mathrm{H}}(x_1,\ldots,x_m)\right)=\frac{\delta_{m,n}}{a_1!\cdots a_n!}\xi_1^{\otimes a_1}\cdots\xi_n^{\otimes a_n}\quad i=(a_1,\ldots,a_n)\,, \tag{45}$$

and extended to all of $\mathcal{F}(M,g)$ by linearity. Here, ξ_i are the Riemannian normal coordinates of x_i relative to x, identified with vectors in T_xM. These functionals satisfy the analog of eq. (44) for the linear fields defined above. Since the interacting fields $\mathcal{O}_i(x)_I$ are given by formal power series whose zeroth order is the linear field expression (see the Bogoliubov formula (5)), it follows that the action of the linear field functionals on an interacting field is of the form

$$\Psi^i_x(\mathcal{O}_j(x)_I)=\delta^i{}_j\,\mathrm{id}_{\mathbf{E}_i}+A^i{}_j(x)\,, \tag{46}$$

where $A^i{}_j$ is the endomorphism in $\mathrm{End}(\mathbf{E}_j,\mathbf{E}_i)$ that arises from the higher perturbative contributions to the interacting field, see (5), and is given by

$$A^i{}_j(x)=\sum_{n\geq 1}\frac{\mathrm{i}^n}{n!}\Psi^i_x\left(\mathbf{R}_n(\mathcal{O}_j(x);I^{\otimes n})\right)\,. \tag{47}$$

Consequently, using the standard geometric series for the inverse of a linear operator of the form $\mathbb{1}+L$ and writing out explicitly the above formula for $A^i{}_j(x)$, we find that the functional defined by the following series is a solution to the equation (44):

$$\begin{aligned}\Psi^i(F)_I=&\sum_{k=0}^{\infty}(-1)^k\sum_{m_l\geq 1}\frac{\mathrm{i}^{m_1+\cdots+m_k}}{m_1!\cdots m_k!}\\ &\Psi^i\left(\mathbf{R}_{m_1}(\mathcal{O}_{j_1};I^{\otimes m_1})\right)\Psi^{j_1}\left(\mathbf{R}_{m_2}(\mathcal{O}_{j_2};I^{\otimes m_2})\right)\cdots\Psi^{j_k}(\mathbf{R}_{m_k}\left(\mathcal{O}_{j_{k+1}};I^{\otimes m_k}\right))\ \Psi^{j_{k+1}}\big(F\big)\,.\end{aligned} \tag{48}$$

Here, $m=\sum m_l$ is the perturbation order of an individual term, and the sums over j_l are carried out to order $[j_l]\leq\Delta$. Thus, for each fixed m, the sum over k has only a finite number of terms, and the resulting expression is a well-defined functional on formal power series, valued in formal power series. Furthermore, all functionals Ψ^{j_k} and all operators $\mathcal{O}_{j_k}$ appearing on the right side are taken at a reference point x. We now define the operator product coefficients by formula (43) in terms of the functionals $\Psi^i(\,.\,)_I$. Writing

out all terms explicitly, the interacting OPE-coefficients are thus given by

$$C_{i_1\dots i_n}{}^{j}(x_1,\dots,x_n)_I \equiv \sum_{k=0}^{\infty}(-1)^k \sum_{m_i\geq 1} \frac{\mathrm{i}^{m_1+\dots+m_k}}{m_1!\cdots m_k!} \sum_{[l_j]\leq\Delta}$$
$$\Psi^j\left(\mathbf{R}_{m_1}(\mathcal{O}_{l_1}(x_n);I^{\otimes m_1})\right)\Psi^{l_1}\left(\mathbf{R}_{m_2}(\mathcal{O}_{l_2}(x_n);I^{\otimes m_2})\right)\cdots\Psi^{l_k}(\mathbf{R}_{m_k}\left(\mathcal{O}_{l_{k+1}}(x_n);I^{\otimes m_k}\right)\right)$$
$$\times\sum_{n_i\geq 0}\Psi^{l_{k+1}}\left(\prod_{r=1}^{n}\frac{\mathrm{i}^{n_r}}{n_r!}\mathbf{R}_{n_i}(\mathcal{O}_{i_r}(x_i);I^{\otimes n_r})\right), \quad (49)$$

where all local functionals Ψ^l refer to the point x_n. In order to make this formula well-defined, it is necessary to assume that the support of the cutoff function χ implicit in I is small enough so that the standard functionals are defined on the corresponding retarded product. However, this is no real restriction, because the OPE is an asymptotic short distance expansion, and we will later show that the coefficients do not depend on the particular choice of χ asymptotically.

We claim that the coefficients C_I satisfy an OPE:

Theorem 1. Let the interaction $I=\int\mathcal{L}\,d\mu$ be renormalizable, i.e., $[\mathcal{L}]\leq 4$. For a given tree $\mathcal{T}$, $\delta\geq 0$, and given $i_1,\dots,i_n\in\mathcal{I}$, let

$$\Delta=\delta+\left(\sum_{j=1}^{n}[i_j]\right)\cdot\mathrm{depth}(\mathcal{T}), \quad (50)$$

and define the OPE coefficients $C_{i_1\dots i_n}{}^{k}$ by eq. (49). Then the OPE holds:

$$\mathcal{O}_{i_1}(x_1)_I\mathcal{O}_{i_2}(x_2)_I\cdots\mathcal{O}_{i_n}(x_n)_I\sim_{\mathcal{T},\delta}\sum_{[k]\leq\Delta}C_{i_1i_2\dots i_n}{}^{k}(x_1,x_2,\dots,x_n)_I\,\mathcal{O}_k(x_n)_I\,. \quad (51)$$

Remarks: 1) The theorem is false for non-renormalizable interactions.
2) Since the topology on $\mathcal{P}\otimes\mathcal{F}(M,g)$ of which the interacting fields are elements is generated by a set of seminorms associated with functionals including the Hadamard states, it follows that the OPE will continue to hold on the factor algebra obtained by dividing by the Klein-Gordon equation, in the sense of expectation values in Hadamard states, to arbitrary orders in perturbation theory.

Proof: Let $\mathcal{N}_I$ be defined as the remainder in the OPE, i.e., the left side of (51) minus the right side. We need to prove that $\varepsilon^{-\delta}\Phi(\mathcal{N}_I\circ\psi_{\mathcal{T}}(\varepsilon))$ tends to 0 in the sense of distributions as $\varepsilon\to 0$. The analysis of this limit is easiest in the case when $\mathcal{T}$ is the tree $\mathcal{T}=\{S_0,S_1,\dots,S_n\}$ with one root $S_0=\{1,\dots,n\}$ and n leaves $S_i=\{i\}$. Then $depth(\mathcal{T})=1$, and $\psi_{\mathcal{T}}(\varepsilon)$ is the map that scales the Riemannian normal coordinates of the points

$x_1, \ldots, x_{n-1} \in U$ relative to $y = x_n$ by ε, where U is a convex normal neighborhood of y. Thus, taking $\varepsilon = 2^{-N}$, we must show that

$$2^{\delta N} \Phi(\mathcal{N}_I(2^{-N} x_1, \ldots, 2^{-N} x_n)) \quad \text{as } N \to \infty, \tag{52}$$

in the sense of distributions valued in $\mathcal{P}$, i.e., to any order in perturbation theory. In the above expression, and in the remainder of this proof, points x_i have been identified with their Riemannian normal coordinates around $y = x_n$. In order to analyze the above expression, it is necessary to perform several intermediate decompositions of $\mathcal{N}_I$, and we now explain how this is done.

We first decompose $\mathcal{N}_I$ into contributions from the different orders in perturbation theory. The ring $\mathcal{P}$ of formal power series in the couplings κ_i contained in the interaction $I = \int \mathcal{L} d\mu$ can be decomposed into a direct sum

$$\mathcal{P} = \bigoplus_k \mathcal{P}_{(k)}, \quad \mathcal{P}_{(k)} = \text{Eigenspace of } \sum \kappa_i \, d/d\kappa_i \text{ for eigenvalue } k, \tag{53}$$

where the k-th summand corresponds to the k-th order in perturbation theory. Accordingly, $\mathcal{N}_I$ may be decomposed as $\sum \mathcal{N}_{(k)}$ into contributions from the various orders in perturbation theory, and likewise $C_I = \sum C_{(k)}$ etc. Using the Bogoliubov formula, the k-th order perturbative contribution to $\mathcal{N}_I$ can be written in the form

$$\begin{aligned} \mathcal{N}_{(k)}(x_1, \ldots, x_n) &= \sum_{k_1 + \cdots + k_n = k} \frac{\mathrm{i}^k}{k_1! \cdots k_n!} \prod_j \mathbf{R}_{k_j}(\mathcal{O}(x_j), I^{\otimes k_j}) \\ &- \sum_{p=0}^{k} C_{(p)}(x_1, \ldots, x_n) \mathbf{R}_{k-p}(\mathcal{O}(x_n), I^{\otimes (k-p)}) \,, \end{aligned} \tag{54}$$

where the labels on $\mathcal{O}$ indicating the field species have been omitted to lighten the notation. Let $A_{(k)}$ be defined as $\mathcal{N}_{(k)}$, but with the k-th order OPE coefficient $C_{(k)}$ omitted. Then, using the definition of the OPE-coefficients, it can be seen that

$$C_{i_1 \ldots i_n}{}^j(x_1, \ldots, x_n)_{(k)} = \Psi^j(A_{i_1 \ldots i_n}(x_1, \ldots, x_n)_{(k)}) \tag{55}$$

and that $\mathcal{N}_{(k)} = A_{(k)} - \sum_{[j] \leq \Delta} \Psi^j(A_{(k)}) \mathcal{O}_j$, where Ψ^j are the free field reference functionals at point x_n and where $\mathcal{O}_j$ are the free field Wick powers taken at point x_n. Thus, the expectation value of the scaled, k-th order perturbative contribution to the remainder is given by

$$\begin{aligned} \Phi(\mathcal{N}_{(k)}(2^{-N} x_1, \ldots, 2^{-N} x_n)) &= \Phi(A_{(k)}(2^{-N} x_1, \ldots, 2^{-N} x_n)) \\ &- \sum_{[j] \leq \Delta} \Psi^j(A_{(k)}(2^{-N} x_1, \ldots, 2^{-N} x_n)) \Phi(\mathcal{O}_j(2^{-N} x_n)) \,, \end{aligned} \tag{56}$$

The right side of this equation is schematically of the form $\Phi(F) - \sum_k \Psi_y^k(F)\Phi(\mathcal{O}_k(y))$, and for such expressions we will now write down an expression which will be useful to analyze the limit $N \to \infty$ of eq. (56). To derive this expression, perform a Taylor expansion with remainder about $(y, \ldots, y) \in U^m$ of the m-th locally normal ordered product (20),

$$:\prod_{i=1}^m \phi(\xi_i):_{\mathrm{H}} - \sum_{|\alpha_1|+\cdots+|\alpha_m|\leq\rho} \frac{1}{\alpha_1!\cdots\alpha_m!} : \prod_{i=1}^m \xi_i^{\alpha_i}\partial^{\alpha_i}\phi(0):_{\mathrm{H}}$$
$$= \frac{1}{\rho!}\int_0^1 (1-t)^\rho\, \partial_t^{\rho+1} \,:\prod_{i=1}^m \phi(t\xi_i):_{\mathrm{H}}\, dt\,. \quad (57)$$

Here, the ξ_i denote Riemannian normal coordinates around y and are identified with points in $\mathbb{R}^4$, $\alpha_i \in \mathbb{N}_0^4$ is a multiindex, and quantities like $|\alpha_i|$ are defined using standard multiindex conventions. As explained above, any element $F \in \mathcal{F}(M,g)$ supported in U may be written as a linear combination of expressions which consist of distributions u_m supported in U^m satisfying the wave front condition (9), integrated with locally normal ordered products $:\phi^{\otimes m}:_{\mathrm{H}}$. If we apply a Hadamard state Φ to such an expression F, use the above Taylor series with remainder, and use the definition (45) for the standard functionals, then we get the following equation:

$$\Phi(F) - \sum_{[k]<\Delta} \Psi_y^k(F)\,\Phi(\mathcal{O}_k(y))$$
$$= \sum_m \sum_{|\alpha|=\Delta-m+1} \frac{1}{(\Delta-m)!}\int_{M^m} u_m(\xi_1,\ldots,\xi_m)\xi_1^{\alpha_1}\cdots\xi_m^{\alpha_m}$$
$$\int_0^1 (1-t)^{\Delta-m}\Phi\left(:\partial_{\alpha_1}\phi(t\xi_1)\cdots\partial_{\alpha_m}(t\xi_m):_{\mathrm{H}}\right) dt \bigwedge_{i=1}^m d\mu(\xi_i)\,. \quad (58)$$

The key point to note about this identity is that there are now factors of $\xi_i^{\alpha_i}$ on the right side, which will work in our advantage when the points ξ_i are scaled by a small factor. On the other hand, the normal ordered expectation values in the second line are smooth (here we are using the assumption that Φ is Hadamard), and so will not cause any trouble for such a scaling. We will now prove that (52) holds in the sense of distributions by exploiting this identity for $F = A_{(k)}$ in eq. (56). However, before we efficiently make use of that identity in (56), it is first necessary to rewrite $A_{(k)}$ in a suitable way, and to apply an induction in k.

For this, we recall that the interaction Lagrangian density $\mathcal{L}$ is confined to the convex normal neighborhood U since we are taking the couplings to be $\kappa_i(x) = \kappa_i\chi(x)$ with χ a smooth cutoff function that is supported in U. We now "slice up" the support of $\mathcal{L}$ into contributions from different "shells" in U that are centered around $y = x_n$, and that have thickness 2^{-j}, where $j = 1, \ldots, N$. For this, we choose a compactly supported function ϑ

that is 1 on U, and we set

$$\vartheta_j(x) = \vartheta(2^j x)\,. \tag{59}$$

Then $\mathcal{L}$ may be decomposed as

$$\mathcal{L} = \vartheta_N \mathcal{L} + \sum_{j=1}^{N} (\vartheta_{j-1} - \vartheta_j)\mathcal{L}\,. \tag{60}$$

Each term in the sum is supported in a slice of thickness 2^{-j}, see the following figure on p. 24. The key step is now to rewrite an interacting field quantity in a way that reflects the subdivision of the interaction region U into these slices. For this, we note that if V_j is the unitary in (31) relating the interacting field with interactions $I_j = \int \vartheta_j \mathcal{L} d\mu$ and $I_{j-1} = \int \vartheta_{j-1}\mathcal{L} d\mu$, we have

$$\mathcal{O}_I(2^{-N}x) = V_1 V_2 \cdots V_N \mathcal{O}_{I_N}(2^{-N}x)(V_1 V_2 \cdots V_N)^{-1}\,, \tag{61}$$

for all $x \in U$. Explicitly, V_j is given in terms of the relative S-matrix [4],

$$V_j = S_{\int \vartheta_j \mathcal{L}}\left(\int \rho_j \mathcal{L}\right) = \sum_k S_{\int \vartheta_j \mathcal{L}}\left(\int \rho_j \mathcal{L}\right)_{(k)}\,. \tag{62}$$

Here, ρ is any smooth function of compact support in U with the property that $\rho(x) = 0$ for all $x \in J^+(\mathrm{supp}(\vartheta_1))$ and $\rho(x) = \vartheta_0(x) - \vartheta_1(x)$ for all $x \in J^-(\mathrm{supp}(\vartheta_1))$, and $\rho_j(x) = \rho(2^j x)$. Each term in this expansion can in turn be written in terms of retarded products [4]. Substituting the equation (61) into the formula for the remainder, and expanding in a perturbation expansion, we get the following identity:

$$\begin{aligned} A_{(k)}(2^{-N}x_1, \ldots, 2^{-N}x_n)_I &= A_{(k)}(2^{-N}x_1, \ldots, 2^{-N}x_n)_{I_N} \\ &+ \sum_{p=0}^{k-1} \sum_{\substack{k-p = k_1 + \cdots + k_r \\ + l_1 + \cdots + l_q}} \prod_{0<\alpha_1 \cdots < \alpha_r < N} S_{\int \vartheta_{\alpha_j} \mathcal{L}}\left(\int \rho_{\alpha_j} \mathcal{L}\right)_{(k_j)} \\ &\cdot \mathcal{N}_{(p)}(2^{-N}x_1, \ldots, 2^{-N}x_n)_{I_N} \\ &\cdot \prod_{0>\beta_1 \cdots > \beta_q > N} S_{\int \vartheta_{\beta_i} \mathcal{L}}\left(\int \rho_{\beta_i} \mathcal{L}\right)^*_{(l_i)}\,. \end{aligned} \tag{63}$$

This complicated identity has the following structure. The sum on the right side is by definition only for p such that $p < k$, meaning that the terms under the sum only contain the remainder in the OPE up to $(k-1)$-th order in perturbation theory. This will enable us to use an inductive procedure to estimate the k-th order perturbative contribution to the remainder by the lower order contribution. The first term on the right side, $A_{(k)}(\ldots)_{I_N}$, is identical in nature with $A_{(k)}(\ldots)_I$, with only exception that all the retarded products

implicit in its definition are now computed with respect to the interaction $I_N = \int \vartheta_N \mathcal{L} d\mu$ which is supported only in a small ball of radius 2^{-N} around $y = x_n$. This will enable us to use a scaling argument to estimate this term.

We now explain more precisely how the decomposition of $A_{(k)}$ given in (63) will make it possible to analyze the scaling behavior of the k-th order remainder in the OPE. For this, we take (63), and substitute it into eq. (56). This gives an expression for $\Phi(\mathcal{N}_{(k)}(\dots)_I)$, and to each term in this expression, we can apply eq. (58). Consider first the term arising from the first term on the right side of eq. (63). That term makes a contribution to $\Phi(\mathcal{N}_{(k)}(\dots)_I)$ of the form

$$\Phi(A_{(k)}(2^{-N}x_1, \dots, 2^{-N}x_n)_{I_N}) - \sum_{[j] \leq \Delta} \Psi^j(A_{(k)}(2^{-N}x_1, \dots, 2^{-N}x_n)_{I_N}) \Phi(\mathcal{O}_j(2^{-N}x_n)_{I_N}) \,. \tag{64}$$

We must now substitute the expression for $A_{(k)}(\dots)_{I_N}$. For this, we use that, on account of the Bogoliubov formula, the scaled interacting field with interaction I_N is given by

$$\mathcal{O}(2^{-N}x)_{I_N} = \sum_k \frac{2^{-4kN} \mathrm{i}^k}{k!} \int_{U^k} \mathbf{R}_k \left(\mathcal{O}(2^{-N}x), \bigotimes_{i=1}^k \mathcal{L}(2^{-N}y_i) \right) \prod \vartheta_0(y_k) d\mu_N(y_k) \,, \tag{65}$$

where we have performed the change of integration variables $y_i \to 2^N y_i$, and where $d\mu_N$ is 2^{4N}-times the pull-back of $d\mu$ by the inverse of this map, which is smooth as $N \to \infty$. If we now also use the Wick expansion of the retarded products (r3), along with the scaling degree property (r4) and combine the result with eq. (58), then we obtain that eq. (64) scales as $2^{-N(\delta+1)}$, as desired.

Now we must take the second term on the right side in (63), substitute it into (56), and then analyze its scaling using (58). To do this, we must now proceed iteratively, in the order in perturbation theory k. For $k = 0$, there is nothing to show. For $k > 0$, we then inductively know the scaling (52) up to order $k-1$, which enables us to estimate the remainders $\mathcal{N}_{(p)}, p < k$ in the terms in the sum on the right side of (63). More precisely, we may inductively assume that the p-th order remainder ($p < k$) has the structure

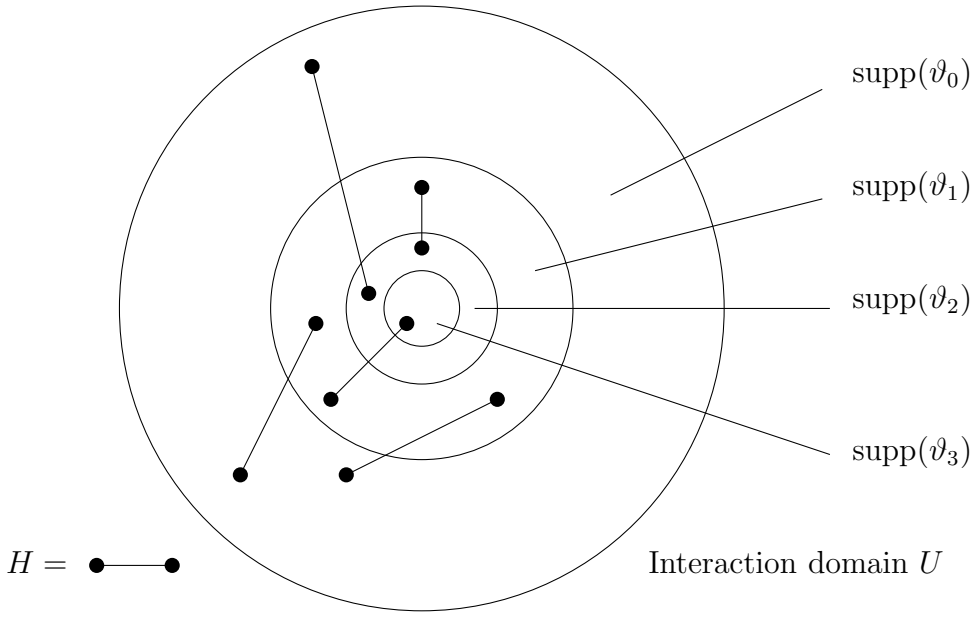
$$\begin{aligned} \mathcal{N}_{(p)}(2^{-N}x_1, \dots, 2^{-N}x_n)_{I_N} = \sum_m 2^{-N(4m+\Delta-m+1)} \sum_{|\alpha|=\Delta-m+1} \frac{1}{(\Delta-m)!} \\ \cdot \int_{M^m} n_m(2^{-N}y_1, \dots, 2^{-N}y_m, 2^{-N}x_1, \dots, 2^{-N}x_n) y_1^{\alpha_1} \cdots y_m^{\alpha_m} \\ \int_0^1 (1-t)^{\Delta-m} \; :\partial_{\alpha_1}\phi(t2^{-N}y_1) \cdots \partial_{\alpha_m}\phi(t2^{-N}y_m):_{\mathrm{H}} \, dt \bigwedge_{i=1}^m d\mu_N(y_i) \,, \end{aligned} \tag{66}$$

where n_m are the coefficients in a Wick-expansion of $\mathcal{N}_{(p)}(x_1, \dots, x_n)_I$ (note that we have also performed a change of integration variables $y_i \to 2^N y_i$). Using the scaling (r3) and

the fact that all terms $\mathcal{N}_{(p)}(\dots)_I$ may be written in terms of retarded products by means of the Bogoliubov formula, one can see that

$$2^{-N([j_1]+\cdots+[j_n]+m[\mathcal{L}]-m)} n_m(2^{-N}y_1, \dots, 2^{-N}y_m, 2^{-N}x_1, \dots, 2^{-N}x_n) \to 0 \quad \text{as } N \to \infty. \quad (67)$$

One now has to take formula (66), and substitute it into the sum on the right side of (63). From the product of $\mathcal{N}_{(p)}(\dots)_{I_N}$ with the relative S-matrices there arise terms which blow up as $N \to \infty$, and so these terms have to be carefully controlled. To understand in detail what type of diverging terms can arise, we must write out the explicit formula for the relative S-matrices in terms of retarded products. Then we must write each retarded product in a Wick expansion (r4), and perform the products using Wick's theorem (10), with ω replaced by H. Then we get a collection of terms, each of which is a product of H, r, n_m and a locally normal ordered Wick power. These terms are evaluated on a set of spacetime arguments which are scaled by $2^{-N}, 2^{-\alpha_i}$, or $2^{-\beta_j}$, and which are integrated against the compactly supported smooth functions ϑ or ρ. The arguments scaled by $2^{-\alpha_j}$ arise from points in the interaction domain U within the α_j-th slice, the arguments scaled by $2^{-\beta_j}$ arise from points in the interaction domain U within the β_j-th slice, and the arguments scaled by 2^N correspond to the scaled arguments $2^{-N}x_i$. More precisely, when we use Wick's theorem to perform the products in the second term in (63), there arise "contractions" between points in the α_i-th and β_j-th slice, indicated by lines in the following figure:



Each such contraction is associated with a factor $H(2^{-\alpha_j}y_1, 2^{-\beta_i}y_2)$ (or a derivative thereof), which, using the explicit form of the Hadamard parametrix H, is seen to scale as $2^{2min(\alpha_i,\alpha_j)}$, in the sense of the scaling degree of a distribution (with a correspondingly

larger power when derivatives are present). Furthermore, the scaling of the retarded products in a term in eq. (63) associated with the i-th slice may also be controlled. Namely, using the Wick expansion (r3), we see that a retarded product associated with the i-th slice contributes factors of $r(2^{-i}y_1, \ldots, 2^{-i}y_l)$, the scaling power of which may then be controlled using (r4). Finally, the scaling of $n_m(2^{-N}x_1, \ldots, 2^{-N}y_m)$ is controlled by eq. (67). Thus, the rate at which the terms in the sum on the right side of (63) blow up can be controlled. We finally need to substitute each such term into eq. (56), and use (58). If we carefully keep track of all the scaling powers, then we find that a typical term contributing to $\mathcal{N}_{(k)}$ arising from these substitutions has the scaling power

$$2^{N(-\delta-4k-1)+[\mathcal{L}]\sum_j k_j\alpha_j+[\mathcal{L}]\sum_i l_i\beta_i} \tag{68}$$

Using a geometric series, the sum of such terms is estimated by

$$\sum_{p=0}^{k-1} \sum_{\substack{k-p=\, k_1+\cdots+k_r \\ +l_1+\cdots+l_q}} \sum_{\substack{1<\alpha_1\cdots<\alpha_r<N \\ 1<\beta_1\cdots<\beta_q<N}} 2^{N(-\delta-4k-1)+[\mathcal{L}]\sum_j k_j\alpha_j+[\mathcal{L}]\sum_i l_i\beta_i}$$

$$\leq \text{const.}\, 2^{-N(\delta+1-([\mathcal{L}]-4)k)} \leq \text{const.}\, 2^{-N(\delta+1)}, \tag{69}$$

where we have used in the last step that the interaction is renormalizable, $[\mathcal{L}] \leq 4$. Thus, the total scaling of the sum of terms in $\mathcal{N}_{(k)}(2^{-N}x_1, \ldots, 2^{-N}x_n)$ is given by $2^{-N(\delta+1)}$, which implies the convergence of (52). On the other hand, for non-renormalizable interactions, we would not get convergence.

The analysis for a general tree $\mathcal{T}$ is in principle not very different from the one just given. For a general tree, the minimum distance between points in a scaled configuration $\vec{x}(2^{-N})$ is of order $2^{-depth(\mathcal{T})N}$, and not just 2^{-N} as in the simple tree studied above. This implies that the scaling of the corresponding quantities in the perturbative expansions is different. One now has to go through the above steps again and take that different scaling into account. If this is done, then the result claimed in the theorem is obtained. □

4 Properties of the OPE coefficients

We would now like to establish a number of important general properties of the OPE coefficients defined in the previous section. These properties are

1. Microlocal spectrum condition.
2. Locality and covariance.
3. Renormalization group.
4. Associativity.

5. Scaling expansion.

Except for the last one, these properties were postulated as axioms in the paper [25], so our present work can be viewed as a confirmation of [25].

We first establish the microlocal spectrum condition. With the graph theoretical notation $\mathcal{G}_{n,m}$ introduced in section 2, let us define the following subset of the cotangent space T^*M^n:

$$\begin{aligned}\Gamma_{n,m}(M,g) \;=\; \Big\{ & (x_1,k_1;\ldots;x_n,k_n) \in T^*M^n \setminus \{0\};\ \exists \text{ decorated graph } G(\vec{x},\vec{y},\vec{p}) \in \mathcal{G}_{m,n} \\ & \text{such that } k_i = \sum_{e:s(e)=x_i} p_e - \sum_{e:t(e)=x_i} p_e \text{ for all } x_i \text{ and} \\ & \text{such that } 0 = \sum_{e:s(e)=y_i} p_e - \sum_{e:t(e)=y_i} p_e \text{ for all } y_i \text{ and} \\ & \text{such that } y_i \in J^+(\{x_1,\ldots,x_n\}) \cap J^-(\{x_1,\ldots,x_n\}) \text{ for all } 1 \leq i \leq m, \Big\}. \end{aligned} \tag{70}$$

The microlocal spectrum condition for the OPE-coefficients is statement that

$$\mathrm{WF}(C) \upharpoonright U_n \subset \overline{\bigcup_{m\geq 0} \Gamma_{m,n}}\,. \tag{71}$$

where U_n is some neighborhood of $\{(x,x,\ldots,x) \in M^n\}$, and where "WF" is the wave front set of a distribution [27].

The microlocal condition in the above form (71) is similar in nature to a condition that was obtained by [3] for the n-point correlation functions of Wick powers in Hadamard states in the context of linear field theory. The difference to the above condition is that also interaction vertices are now allowed, which were not considered in [3]. These interaction vertices correspond to the contributions $m \geq 1$ in (71) and genuinely weaken the bound on the wave front set relative to the linear case (for $n \geq 4$). The interaction vertices arise from the non-linear interactions present in the theory. As we will see, the maximum valence of the interaction vertices allowed in $\mathrm{WF}(C)$ is equal to the maximum power of the field ϕ that appears in the interaction Lagrangian $\mathcal{L}$. Since we restrict ourselves to renormalizable interactions in 4 spacetime dimensions, that maximum valence is equal to 4. Note, however, that the wave front condition (71) is only an upper bound, and does not say whether interaction vertices will actually contribute to $\mathrm{WF}(C)$ or not. We have checked this for the OPE-coefficient in front of the identity operator in the expansion of the product $\phi(x_1)_I \cdots \phi(x_4)_I$ of four interacting fields, to first order in perturbation theory in Minkowski space, where a contribution from an interaction vertex in $\Gamma_{4,1}$ would be allowed according to the above estimate (71). Using our definition (49) of C and using

the integrals in [9], we found in this example that such a contribution is actually absent from WF(C). Hence, the estimate (71) is not sharp.

Let us now prove the microlocal condition (71). By eq. (49), the microlocal spectrum condition will follow if we can show that if

$$u_n(x_1,\ldots,x_n) = \Psi^j\left(\prod_{r=1}^{n} \mathbf{R}_{n_r}(\mathcal{O}_{i_r}(x_r); I^{\otimes n_r})\right), \tag{72}$$

then $\mathrm{WF}(u_n) \subset \cup_m \Gamma_{n,m}$. To prove this statement, we expand the retarded products as in eq. (27), then multiply them using the Wick expansion formula (10), and finally apply the functional Ψ^j. The result will be sum of expressions each of which is a product of r's, of H's and expectation values in Ψ^j of locally normal ordered expressions, which are integrated over interaction vertices against the smooth test function χ of compact support appearing as infrared cutoff in the Lagrange density, $\mathcal{L}$. The expectation value in Ψ^j of any locally normal ordered expression is smooth, the wave front set of the r's is given above in (r6), see (30), while the wave front property of H is $\mathrm{WF}(H) = \{(y_1,k_1;y_2,k_2); k_1 = p, k_2 = -p,\ p \in \partial V_+^*\}$, where p is a coparallel, cotangent vector field along a null geodesic (edge) joining y_1, y_2. We now combine these facts using the wave-front set calculus of Hörmander, by which we mean the following theorems about the behavior of the wave front set under the operations of smoothing, and products [27]: Let X, Y be manifolds (in our applications, they are Cartesian powers of M). If $K \in \mathcal{D}'(X \times Y)$ is a distribution and $f \in \mathcal{D}(Y)$ a smooth test function, then the distribution $u(x) = \int_Y K(x,y) f(y) d\mu(y)$ has wave front set

$$\mathrm{WF}\,(u) \subset \{(x,k) \in T^*X;\ (x,k;y,0) \in \mathrm{WF}(K)\} \tag{73}$$

Secondly, let $u, v \in \mathcal{D}'(X)$ so that $[\mathrm{WF}(u) + \mathrm{WF}(v)] \cap \{0\} = \emptyset$. Then the distributional product uv is defined and has wave front set

$$\mathrm{WF}(uv) \subset \{(x,k+p) \in T^*X;\ (x,k) \in \mathrm{WF}(u) \cup \{0\},\ (x,p) \in \mathrm{WF}(v) \cup \{0\}\}\,. \tag{74}$$

Applying these rules to the above products of r's and H's, we essentially obtain that $\mathrm{WF}(u_n)$ is a subset of $\cup_m \Gamma_{n,m}$. For example, the momentum conservation rule in the third line of eq. (70) follows from the additive and smoothing properties (73), (74) combined with the fact that we are integrating the interaction vertices against the smooth testfunction, χ, of compact support. Similarly, we obtain the second line from the additive property (74). Finally, we need to prove the support restriction on the interaction vertices in the fourth line of (70). For this, we note that the contribution to the wave front set of u_n from the interaction vertices y_k arises only from points that are in the support of the interaction, I, i.e., in the support of χ. Let U be an arbitrary small neighborhood of $J^-(\{x_1,\ldots,x_n\}) \cap J^+(\{x_1,\ldots,x_n\})$, and let χ' be a cutoff function supported in U.

Then $\chi - \chi'$ is supported outside of the domain of dependence $D(\{x_1, \ldots, x_n\})$, and so there exists by (31) a unitary V such that

$$\prod_k \mathcal{O}_{i_k}(x_k)_{I'} = V \left\{ \prod_k \mathcal{O}_{i_k}(x_k)_I \right\} V^* \,. \tag{75}$$

Thus, because of (43), we see that changing χ to χ is equivalent to changing the standard functionals from $\Psi^j(\ldots)$ to $\Psi^j(V \ldots V^*)$. We claim that this would not, however, change our above wave front argument. Indeed, the only property of the functionals that was used in the above wave front set argument was that the expectation values of locally normal ordered expressions in Ψ^j are smooth. This does not change if we change the standard functionals from $\Psi^j(\ldots)$ to $\Psi^j(V \ldots V^*)$. Consequently, we have shown that contributions to (70) arise only from interaction vertices y_k in U. Since U was an arbitrarily small neighborhood of $J^-(\{x_1, \ldots, x_n\}) \cap J^+(\{x_1, \ldots, x_n\})$, the support restriction in the last line of (70) follows.

We next show that the OPE coefficients have the following local and covariance property: Let $f : M \to M'$ be a causality preserving isometric embedding, let C_I respectively $C'_{I'}$ be the OPE coefficients on the respective spacetimes, and let $\delta > 0$ be given. Finally, assume that there are open neighborhoods $U \subset M$ and $U' \subset M'$ with $f(U) \subset U'$ where the cutoff functions χ respectively χ' implicit in the interactions I and I' are equal to 1. Then, supposing that Δ in eq. (44) is chosen as in Theorem 1, we have on U

$$f^* C'_{I'} \sim_{\mathcal{T},\delta} C_I \quad \text{for all trees } \mathcal{T}. \tag{76}$$

This condition essentially follows from the fact that the interacting fields are local and covariant, in the sense of (13), which follows in turn from the fact that the individual terms in the perturbation expansion of the interacting fields are local and covariant. However, a complication arises from the fact that the algebra embedding α_f in (13) is not simply given in terms of the corresponding free field homomorphism, but is more complicated [23].

Instead of taking into account the more complicated definition of α_f at the interacting level, one can also more directly prove (76). For this, we note that, if the cutoff function χ' on M' were such that $f^*\chi' = \chi$, then we would have equality in (76), because the retarded products and standard functionals which are the ingredients in the definition of C_I have a local and covariant dependence simultaneously on *both* χ and the metric g implicit in I, by property (r5). Thus, it is sufficient to show that C_I is essentially independent of the cutoff function χ. In other words, if χ and χ' are two cutoff functions (on the *same* spacetime) which are equal to 1 on U, and if C_I and $C_{I'}$ are the corresponding OPE coefficients, then we must show that

$$C_I \sim_{\delta,\mathcal{T}} C_{I'} \tag{77}$$

holds on U. To prove this statement, we simply apply the functionals $\Psi(V \ldots V^*)_I$ to the remainder $\mathcal{N}_{I'}$ of the OPE formed using the coefficients $C_{I'}$, where V is the unitary in

eq. (31) relating the interactions I and I'. Then we find $\Psi(V\mathcal{N}_{I'}V^*)_I = C_I - C_{I'}$. Since $\mathcal{N}_{I'}$ is the remainder of the OPE, and since $\Psi(V\dots V^*)_I$ is a Hadamard functional, it follows that $\varepsilon^{-\delta}(C_I - C_{I'}) \circ \psi_{\mathcal{T}}(\varepsilon)$ will go to zero by theorem 1, which is what we needed to show.

In [23], it was shown that the perturbative interacting fields obey a "local covariant renormalization group flow". The construction of this flow involves the consideration of a 1-parameter family $\lambda^2 g$ of conformally rescaled metrics, where $\lambda \in \mathbb{R}_{>0}$, and states how the interacting fields $\mathcal{O}_i(x)_I$ change under such a rescaling. In [25], a simple general argument was given that the existence of such a local covariant renormalization group flow implies a corresponding flow of the OPE coefficients if the theory possesses an OPE in the sense described in the previous section. The key assumption on the nature of the RG made in [25] was that there exists a suitable "basis" of functionals which is in some sense "dual" to the fields. This is the case in perturbation theory, on account of (44). Hence, it follows by the argument of [25] that

$$Z(\lambda)^{i_1}{}_{j_1} \cdots Z(\lambda)^{i_1}{}_{j_1}[{}^t Z(\lambda)^{-1}]_k{}^l C_{i_1\dots i_n}{}^k[M,g]_I \sim_{\mathcal{T},\delta} C_{j_1\dots j_n}{}^l[M,\lambda^2 g]_{I(\lambda)} \tag{78}$$

for all trees $\mathcal{T}$. Here, the prefactors are linear maps (whose construction and properties was described in [23])

$$Z(\lambda)^i{}_j \in \mathrm{End}(\mathbf{E}_j, \mathbf{E}_i)\,, \tag{79}$$

where $\mathbf{E}_i$ is the vector bundle in which the field $\mathcal{O}_i$ lives, and $I(\lambda)$ is an interaction of the same form as I with suitable "running" couplings $\kappa_i(\lambda)$, whose construction was also described in [23].

An associativity property for the OPE coefficients may be formulated as follows (see [25] for details). Let $\vec{x}(\varepsilon)$ be a curve in configuration space representing the merger of the points according to a tree $\mathcal{T}$, that is, $\psi_{\mathcal{T}}(\varepsilon) : \vec{x} \mapsto \vec{x}(\varepsilon)$. In this situation, we should be able to perform the OPE successively, in the hierarchical order represented by the tree, thus leading to some kind of "asymptotic factorization". That is, we should be allowed to first perform the OPE for each subtree, and then successively the OPE's corresponding to the branches relating the subtrees, and so fourth. For example, for the tree $\mathcal{T}$ given in the figure on p. 16, we should be allowed to perform the OPE successively as indicated by the brackets $(\mathcal{O}_1\mathcal{O}_2)(\mathcal{O}_3\mathcal{O}_4)$. To formulate this condition more precisely, we recall the notation $s(e), t(e)$ for the source and targets of an edge, e, in the tree $\mathcal{T}$. Furthermore, for $\vec{x} = (x_1, \dots, x_n) \in M^n$, and for each node $S \in \mathcal{T}$ of the tree, let us set

$$x_S = x_{m(S)}, \quad m(S) = \max\{i : i \in S\}\,. \tag{80}$$

Finally, we consider maps $\vec{i} : \mathcal{T} \to \mathcal{I}$ which associates with every node $S \in \mathcal{T}$ of the tree an element $i_S \in \mathcal{I}$, the index set labelling the fields. With these notations in place, the

associativity property can be stated as follows. Let $\delta > 0$, let $\mathcal{T}$ be a tree, and let M_S^n be the set of all "spacelike configurations" $\vec{x} = (x_1, \ldots, x_n)$

$$M_S^n = \{\vec{x} \in M_0^n; \quad x_i \notin J^+(x_j) \cup J^-(x_j) \text{ forall } i, j\}\,. \tag{81}$$

Then, on M_S^n, we have

$$C_{i_1 \ldots i_n}{}^j(x_1, \ldots, x_n)_I \\ \sim_{\mathcal{T},\delta} \sum_{\vec{i}} \prod_{S \in \mathcal{T}} C_{\{i_{t(e)}; e \text{ such that } s(e)=S\}}{}^{i_S} \left(\{x_{t(e)}; e \text{ such that } s(e) = S\}\right)_I\,, \tag{82}$$

where the sum is over all $\vec{i}$, with the properties that

$$i_{\{k\}} = i_k, \quad k = 1, \ldots, n, \quad i_{\{1,\ldots,n\}} = j, \tag{83}$$

$$\sum_{e: s(e)=S} [i_{t(e)}] < \delta_S \quad \forall S \in \mathcal{T}\,, \tag{84}$$

where $\delta_S > 0$ are chosen sufficiently large. Note that it makes sense to consider the relation $\sim_{\delta,\mathcal{T}}$ with respect to the open subset of spacelike configurations, because a configuration remains spacelike when scaled down by $\psi_{\mathcal{T}}(\varepsilon)$, at least provided the points $\vec{x} \in M_S^n$ are in a sufficiently small neighborhood of the total diagonal, which we assume is the case. The technical reason for restricting the OPE to pairwise spacelike related points is that the OPE coefficients C are smooth on M_S^n, by the microlocal spectrum condition (see eq. (71)), and so convergence in the sense of $\sim_{\delta,\mathcal{T}}$ is more straightforward to study. Furthermore, since the interacting fields commute for spacelike related points, there are no ordering issues when working with the configurations in M_S^n. Also, from a physical viewpoint, the notion of "short distances" is somewhat unclear if lightlike directions are included.

In [25], it is shown that associativity in the above sense is an automatic consequence if the theory also possess a suitable local covariant renormalization group with suitable properties, and if the OPE holds not only for each fixed spacetime and fixed choice of couplings, but instead also uniformly in a suitable sense for smooth families of metrics and couplings (termed "condition (L)" in that paper). As we have already described, the existence of a local covariant renormalization group in perturbation theory was established in [23]. It is a general property of the perturbative renormalization group that $Z^i{}_j(\lambda)$ is given by $\lambda^{-[i]}$ times a polynomial in $\ln \lambda$ at each finite order in perturbation theory [23], and that the running couplings $\kappa_i(\lambda)$ in $I(\lambda)$ have a power law dependence $\lambda^{4-[i]}$, which is modified by polynomials in $\ln \lambda$ at any given order in perturbation theory. These are essentially the properties for the renormalization group required in (L) at the perturbative level, except that the running of couplings $I(\lambda)$ is not exactly smooth at $\lambda = 0$ as required in (L), but instead contain logarithmic terms at each order in perturbation theory. However, the argument given in [25] is insensitive to such logarithmic corrections.

To also establish the desired smooth dependence of the OPE-coefficients under smooth variations of the metric required in (L) at the perturbative level, it is necessary to go through the proof of Theorem 1 for families of spacetimes and corresponding families of states depending smoothly on a parameter in the sense [22] and analyze the behavior of the constructions under variations of the parameter. This can indeed be done, using the smooth dependence of the retarded products under such parameters [22], as well as the techniques and type of arguments employed in the appendix of [24]. However, even though the repetition of these arguments is in principle straightforward, that analysis is quite lengthy and cumbersome, and not very illuminating. It is therefore omitted. Since a perturbative version of condition (L) holds, Theorem 1 of [25] then implies that the associativity property holds on the space M_S^n of pairwise spacelike configurations.

In the remainder of this section we will prove that the OPE coefficients themselves can be expanded for asymptotically small distances in terms of curvature terms and Minkowski distributions in the tangent space (for spacelike related configurations in M_S^n, to which we shall restrict ourselves in the remainder of this section). The construction of this "scaling expansion" involves the Mellin transform, $\mathcal{M}[f,z]$, of a function $f(x)$ defined on $\mathbb{R}_{>0}$ vanishing near infinity, with at most polynomial type singularity [39] as $x \to 0$. It is defined by

$$\mathcal{M}[f,z] = \int_0^\infty x^{\mathrm{i}z-1} f(x)\, dx\,, \tag{85}$$

and is an analytic function of z for sufficiently small $\mathrm{Im}(z) < y_0$ where y_0 depends upon the strength of the singularity of f. The inverse Mellin transform of $F(z) = M[f,z]$ is given by

$$\mathcal{M}^{-1}[F,x] = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty+c}^{+\mathrm{i}\infty+c} x^z F(z)\, dz \tag{86}$$

where the integration contour is to the right of all poles of $F(z)$ in the complex z-plane. The Mellin transform is useful in the context of functions $f(x)$ possessing near $x = 0$ an asymptotic expansion of the form

$$f(x) \sim \sum_p \sum_l a_{p,l} x^{-p} (\ln x)^l\,, \tag{87}$$

where p is bounded from above, and where the sum over l is finite for any p. It can be seen that the Mellin transform of such a function possesses isolated poles at $z = \mathrm{i}p$ in the complex plane, with finite multiplicities. Furthermore, the asymptotic expansion coefficients $a_{p,l}$ are the residues of the Mellin transform, i.e.,

$$a_{p,l} = \frac{1}{l!} \mathrm{Res}_{z=-\mathrm{i}p} \Big\{ (z+\mathrm{i}p)^l \mathcal{M}[f,z] \Big\}\,. \tag{88}$$

We now define, for each tree $\mathcal{T}$, distributions $C_I^{\mathcal{T}}$ that give the desired scaling expansion of the OPE coefficient C_I relative to the scaling function $\psi_{\mathcal{T}}(\varepsilon) : M_0^n \to M_0^n, \vec{x} \mapsto \vec{x}(\varepsilon)$

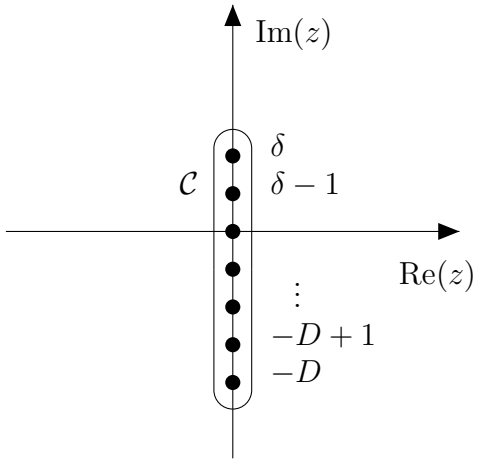
defined above in eq. (39), by extracting the poles of $C_I \circ \psi_{\mathcal{T}}(\varepsilon)$ in ε using the Mellin transform. In order to do this, let $\vec{x} = (x_1, \ldots, x_n) \in M_S^n$ be a spacelike configuration of n points, let $\mathcal{T}$ be a tree, and let $\varepsilon \mapsto \vec{x}(\varepsilon)$ be the corresponding curve in M_S^n. By the microlocal spectrum property, the OPE coefficients are smooth on M_S^n, so we may consider $[C_I \circ \psi_{\mathcal{T}}(\varepsilon)](\vec{x}) = C_I(\vec{x}(\varepsilon))$ as a smooth function in ε at any fixed value of the argument $\vec{x}$. As we will show in the proof of the next theorem, if we fix the parameter $\delta > 0$ in the operator product expansion, this function has an expansion of the form

$$C_I(\vec{x}(\varepsilon)) = \sum_p \sum_l a_{p,l}(\vec{x}) \varepsilon^{-p} (\ln \varepsilon)^l + \ldots \,, \tag{89}$$

near $\varepsilon = 0$, where the dots stand for a remainder vanishing faster than ε^δ. Here, p is in the range from $-\delta$ to $D = \mathrm{depth}(\mathcal{T}) \cdot (-[k] + \sum_j [i_j])$, and the sum over l is finite for each p, at any given order in perturbation theory[7]. Consequently, we can define the Mellin transform[8] of this function in the variable ε

$$\mathcal{M}^{\mathcal{T}}(\vec{x}, z) \equiv \mathcal{M}[C_I \circ \psi_{\mathcal{T}}(\varepsilon), z] = \int_0^\infty C_I(\vec{x}(\varepsilon))\, \varepsilon^{\mathrm{i}z-1}\, d\varepsilon \,, \tag{90}$$

which is now a function of $\vec{x} \in M_S^n$ that is analytic in $z \in \mathbb{C}$ for sufficiently small $\mathrm{Im}(z)$. Furthermore, by the above expansion (89), it is meromorphic on a domain including $\mathrm{Im}(z) \leq \delta$, with poles possibly at $\mathrm{i}\delta, \mathrm{i}(\delta-1), \mathrm{i}(\delta-2), \ldots, -\mathrm{i}D$. Let us now choose a contour $\mathcal{C}$ around these points as illustrated in the figure.



Define

$$C^{\mathcal{T}}(\vec{x})_I \equiv \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{C}} \mathcal{M}^{\mathcal{T}}(\vec{x}, z)\, dz \,. \tag{91}$$

Concerning this function on M_S^n, we have the following theorem.

[7]Note however that the range of l increases with the perturbation order.

[8]To make this expression well defined, we need to arbitrarily cut off the integral for large ε (where the map $\psi_{\mathcal{T}}(\varepsilon)$ is not well-defined anyway). How this cutoff is chosen does not affect the following discussion.

Theorem 2. 1. We have $C_I^{\mathcal{T}} \sim_{\mathcal{T},\delta} C_I$ for spacelike configurations, and therefore

$$\mathcal{O}_{i_1}(x_1)_I \mathcal{O}_{i_2}(x_2)_I \cdots \mathcal{O}_{i_n}(x_n)_I \sim_{\mathcal{T},\delta} \sum_{[k]\leq\Delta} C^{\mathcal{T}}_{i_1 i_2 \ldots i_n}{}^{k}(x_1, x_2, \ldots, x_n)_I \, \mathcal{O}_k(x_n)_I \tag{92}$$

on M_S^n.

2. $C_I^{\mathcal{T}}$ is local and covariant, i.e., if $f : (M, g) \to (M', g')$ is an orientation, causality preserving isometric embedding then $C_I^{\mathcal{T}} = f^* C'^{\mathcal{T}}_{I'}$. In particular, $C_I^{\mathcal{T}}$ does not depend on the choice of cutoff function χ used in the definition of the interacting field.

3. The expression $C_I^{\mathcal{T}}(\vec{x})$ is the sum of residue of $\mathcal{M}^{\mathcal{T}}(z, \vec{x})$ corresponding to the poles in the contour $\mathcal{C}$ in (91),

$$C_I^{\mathcal{T}}(\vec{x}) = \sum_{p\geq-\delta} \mathrm{Res}_{z=-\mathrm{i}p} \left\{ \mathcal{M}^{\mathcal{T}}(\vec{x}, z) \right\} . \tag{93}$$

These have the following form:

$$\mathrm{Res}_{z=-\mathrm{i}p} \left\{ \mathcal{M}^{\mathcal{T}}(\vec{x}, z) \right\} = \sum_a P_a [\nabla_{(\alpha_1} \cdots \nabla_{\alpha_k)} R_{\mu_1\mu_2\mu_3\mu_4}(x_n)] \, W^a(\xi_1, \ldots, \xi_{n-1}) , \tag{94}$$

where P_a is a polynomial in the Riemann tensor and (finitely many) of its covariant derivatives evaluated at x_n, valued in some tensor power of the tangent space $(T_{x_n} M)^{\otimes a}$, while ξ_i are the Riemannian normal coordinates of $x_1, \ldots, x_{n-1}$ around x_n, identified with vectors in $\mathbb{R}^4$ via a tetrad. The sum over a is finite, and each W_a is a Lorentz covariant distribution on $\mathbb{R}^{4(n-1)}$ (defined on spacelike configurations), that is valued in $(\mathbb{R}^4)^{\otimes a}$ (identified with $(T^*_{x_n} M)^{\otimes a}$ via the tetrad), depending polynomially on m^2, α. Thus, for any proper, orthochronous Lorentz transformation Λ, we have

$$W_a(\Lambda\xi_1, \ldots, \Lambda\xi_{n-1}) = D^b{}_a(\Lambda) W_b(\xi_1, \ldots, \xi_{n-1}) , \tag{95}$$

where $D(\Lambda)$ is the corresponding tensor representation.

4. There are Lorentz invariant distributions $V_{a,l}$ such that

$$W_a(\vec{\xi}(\varepsilon)) = \varepsilon^N \left[W_a(\vec{\xi}) + \sum_l (\ln \varepsilon)^l V_{a,l}(\vec{\xi}) \right] , \tag{96}$$

where the sum is only over a finite range of l, at any given, but finite order in perturbation theory.

Remarks: 1) Equations (93), (94) constitute the claimed scaling expansion. That scaling expansion is similar in nature to a corresponding expansion derived in [22] for the short distance behavior of time ordered products. However, note that the scaling expansion above is more general than that derived in [22], because it involves the consideration of more general scaling functions $\psi_{\mathcal{T}}(\varepsilon)$ corresponding to general trees. Also, the Mellin transformation technique was not used in [22].

2) The restriction to spacelike configurations has mainly been made for technical reasons, to avoid technical issues that could arise when taking the Mellin transformation of a distribution. However, we expect that all constructions and properties summarized in the above theorem hold for all configurations, i.e., in the sense of distributions on M^n.

Proof: Let us first argue that the claimed meromorphicity of the Mellin transform of $C(\vec{x}(\varepsilon))_I$ holds, or equivalently, that it has an asymptotic expansion as claimed in eq. (89). For this, we first consider the simple tree $\mathcal{T} = \{S_0, S_1, \ldots, S_n\}$ with one root $S_0 = \{1, \ldots, n\}$ and n leaves $S_i = \{i\}$. The depth of this tree is $depth(\mathcal{T}) = 1$, and $\psi_{\mathcal{T}}(\varepsilon)$ is simply the map which multiplies the Riemann normal coordinates of x_i relative to x_n by ε. To analyze the corresponding scaling of the OPE coefficients, we note that, by the local and covariance property of the OPE coefficients, a rescaling of the arguments x_i is equivalent to changing the metric from g to $s_\varepsilon^* g$, where $s_\varepsilon : M \to M$ is the diffeomorphism that scales the Riemannian coordinates of a point around x_n by ε. Thus, we have have

$$C_I[M, g] \circ \psi_{\mathcal{T}}(\varepsilon) \sim_{\mathcal{T},\delta} C_I[M, s_\varepsilon^* g] \tag{97}$$

for this tree. Next, we use the fact that in perturbation theory, there exists a local and covariant renormalization group [23]. This implies that, up to terms of order ε^δ

$$Z(\varepsilon)^{i_1}{}_{j_1} \cdots Z(\varepsilon)^{i_1}{}_{j_1} [{}^t Z(\varepsilon)^{-1}]_k{}^l C_{i_1 \ldots i_n}{}^k[M, g(\varepsilon)]_{I(\varepsilon)} = C_{j_1 \ldots j_n}{}^l[M, g]_I \circ \psi_{\mathcal{T}}(\varepsilon)\,, \tag{98}$$

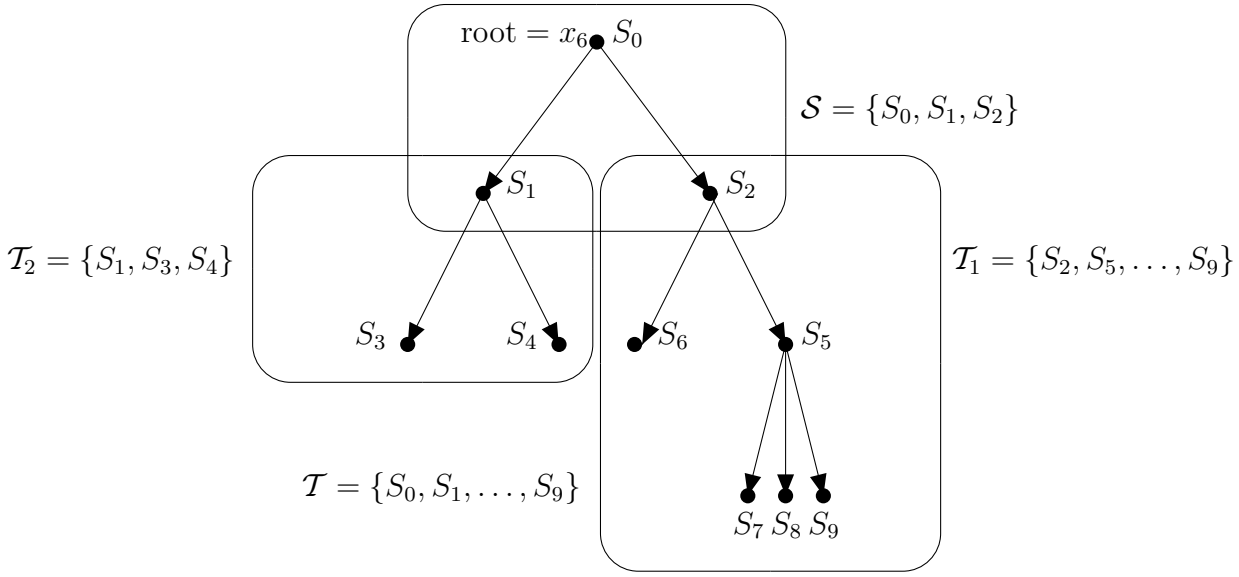
where $g(\varepsilon) = \varepsilon^2 s_\varepsilon^* g$. The point is now that the metric $g(\varepsilon)$ has a smooth dependence upon ε, as may be seen by rewriting it in Riemannian normal coordinates as $g_{\mu\nu}(\varepsilon\xi) d\xi^\mu d\xi^\nu$. The OPE coefficients in turn have a smooth dependence upon smooth variations of the metric, since all the quantities in their definition have this property [22]. The only singular terms (in ε) in the expression on the left side can therefore come from (a) the running couplings in $I(\varepsilon)$ and (b) the $Z(\varepsilon)$-factors. However, by the general analysis of the local renormalization given in [23], these quantities are polynomials in $\varepsilon^{-p}(\ln \varepsilon)^l$ of finite degree to any given order in perturbation theory. Thus, up to terms vanishing faster than ε^δ, the quantity $C_I(\vec{x}(\varepsilon))$ has an expansion of the type (89), as desired.

The argument just given may be generalized to arbitrary trees by an induction in the depth of the tree. For trees of depth one we have just proven the statement. Let us inductively assume that we have proven (89) for trees of depth d. To deal with trees of depth $d+1$, one notices that a tree of depth ≥ 2 can always be decomposed into a tree $\mathcal{S}$ of depth one connected to the root, and trees $\mathcal{T}_1, \ldots, \mathcal{T}_r$ attached to the leaves of $\mathcal{S}$.

Thus, we may write

$$\mathcal{T} = \mathcal{S} \cup \bigcup_{t=1}^{r} \mathcal{T}_t \,, \tag{99}$$

see the figure for an example with $r = 2$.



The key point is now that the map $\psi_{\mathcal{T}}(\varepsilon)$ factorizes by the inductive nature of its definition (39), while the OPE coefficient C_I factorizes by the associativity property (82) under this decomposition of the tree. This gives

$$C_{i_1 \ldots i_n}{}^{k} \circ \psi_{\mathcal{T}}(\varepsilon) \sim_{\mathcal{T},\delta} \left[C_{j_1 \ldots j_r}{}^{k} \cdot \bigotimes_{t=1}^{r} C_{\{i_S;\, S \in \mathrm{Leaves}(\mathcal{T}_t)\}}{}^{j_t} \circ \psi_{\mathcal{T}_t}(\varepsilon) \right] \circ \psi_{\mathcal{S}}(\varepsilon) \,, \tag{100}$$

where we note that, since $\mathcal{S}$ has depth one, the map $\psi_{\mathcal{S}}(\varepsilon)$ is given in terms of the diffeomorphism $s_\varepsilon : M \to M$ which scales the Riemann normal coordinates around x_n of points by ε. On the right side of this expression, we can now apply the induction hypothesis to the expression in brackets, because each of the trees $\mathcal{T}_t$ has depth $\leq d$. Furthermore, since $\psi_{\mathcal{S}}(\varepsilon)$ is given by the diffeomorphism s_ε, we may again apply the general covariance and renormalization group property as we did above to convert the action of s_ε into a smooth change $g(\varepsilon)$ in the metric and $Z(\varepsilon)$-factors depending polynomially on $\varepsilon^{-p}(\ln \varepsilon)^l$. This proves the equation (89).

We next come to the actual proof of the theorem. To prove 1), we need to show that $\varepsilon^{-\delta} (C_I - C_I^{\mathcal{T}}) \circ \psi_{\mathcal{T}}(\varepsilon) \to 0$ as $\varepsilon \to 0$, to any finite but arbitrary order in perturbation

theory. We have, using the definition of $C_I^{\mathcal{T}}$ and the relation $\psi_{\mathcal{T}}(\varepsilon)\circ\psi_{\mathcal{T}}(\varepsilon') = \psi_{\mathcal{T}}(\varepsilon\varepsilon')$

$$\begin{aligned} C_I^{\mathcal{T}}\circ\psi_{\mathcal{T}}(\varepsilon) &= \sum_{p\geq-\delta}\mathrm{Res}_{z=-\mathrm{i}p}\int_0^\infty (C_I\circ\psi_{\mathcal{T}}(\varepsilon'))\circ\psi_{\mathcal{T}}(\varepsilon)\,\varepsilon'^{\mathrm{i}z-1}\,d\varepsilon' \\ &= \sum_{p\geq-\delta}\mathrm{Res}_{z=-\mathrm{i}p}\Big\{\mathrm{e}^{\mathrm{i}z\ln\varepsilon}\,\mathcal{M}^{\mathcal{T}}(\vec{x},z)\Big\} \\ &= \sum_{p\geq-\delta}\varepsilon^{-p}\sum_l\frac{(\ln\varepsilon)^l}{l!}\mathrm{Res}_{z=-\mathrm{i}p}\left\{(z+\mathrm{i}p)^l\mathcal{M}^{\mathcal{T}}(\vec{x},z)\right\} \\ &= \sum_{p\geq-\delta}\sum_l\varepsilon^{-p}(\ln\varepsilon)^l a_{p,l}(\vec{x})\,, \end{aligned} \tag{101}$$

where we have performed a change of integration variables in the second step. Comparing with (89), this formula implies that $C^{\mathcal{T}}(\vec{x}(\varepsilon))_I$ differs from $C(\vec{x}(\varepsilon))_I$ by a term vanishing faster than ε^δ. This proves the assertion 1).

To prove 2), we recall that we have already proven above that, at each order in perturbation theory, $f^*C_{I'}[M',g'] \sim_{\delta,\mathcal{T}} C_I[M,g]$, so the difference between these two terms vanishes faster than ε^δ. This difference term will change the Mellin transform $\mathcal{M}^{\mathcal{T}}(z)$ only by a term that is analytic in a domain including $\mathrm{Im}(z)\leq\delta$, and thus will not contribute to $C_I^{\mathcal{T}}[M,g]$ respectively $C_{I'}^{\mathcal{T}}[M',g']$, because the contour integral of a holomorphic function vanishes.

For 3), consider again the metric $g(\varepsilon) = \varepsilon^2 s_\varepsilon^* g$. Its components in Riemann normal coordinates around x_n have a Taylor expansion of the form $g_{\mu\nu}(\varepsilon\xi) = \sum\varepsilon^N P_N(\nabla^k R_{\alpha_1\alpha_2\alpha_3\alpha_4}(x_n),\xi^\rho)_{\mu\nu}$, where P_N are polynomials. Since the $C_I^{\mathcal{T}}$ are local and covariant by 2), it follows that they can be viewed as functionals of the Riemann tensor and its covariant derivatives at point x (or $\xi=0$) which enters via P_N. Thus, it follows that the N-th ε-derivative is

$$\frac{\partial^N}{\partial\varepsilon^N}C_I^{\mathcal{T}}[g(\varepsilon)](\vec{x})|_{\varepsilon=0} = \sum_{N_1+\cdots+N_r=N} W^{N_1\ldots N_r}(\vec{\xi})\prod_i P_{N_i}[\nabla_{(\alpha_1}\cdots\nabla_{\alpha_k)}R_{\mu_1\mu_2\mu_3\mu_4}(x_n),\xi_j^\nu]\,, \tag{102}$$

where

$$W^{N_1\ldots N_r} = \frac{\partial^r C_I^{\mathcal{T}}}{\partial P_{N_1}\cdots\partial P_{N_r}}\Big|_{g=\eta}. \tag{103}$$

We define W^a and P_a by the above relation, i.e., P_a is the appropriate product of the P_{N_i}, with the polynomial ξ_i-dependence taken out and absorbed in the definition of W^a. To prove the desired relation (94), we must now show that the ε-derivatives of $C_I^{\mathcal{T}}[g(\varepsilon)]$ at $\varepsilon = 0$ vanish for N sufficiently large. This quantity arises from the quantity $C_I[g(\varepsilon)]\circ\psi_{\mathcal{T}}(\varepsilon')$ by taking a Mellin transform in ε', then taking N derivatives with respect to ε at $\varepsilon=0$, and finally extracting the residue in z. It follows that we only need to show that the quantity obtained by taking N derivatives with respect to ε of $C_I[g(\varepsilon)]\circ\psi_{\mathcal{T}}(\varepsilon')$ vanishes

faster than ε'^{δ}, because such a term would not give rise to poles of the Mellin transform in the domain $\mathrm{Im}(z) \leq \delta$. Consider first the case when $\mathcal{T}$ has depth one, so that $\psi_{\mathcal{T}}(\varepsilon')$ is simply a dilation of the Riemann normal coordinates by ε'. As above in eq. (98), by combining the renormalization group and general covariance, the action of such a dilation may be translated into changing $g(\varepsilon)$ to $g(\varepsilon\varepsilon')$, along with a suitable set of $Z(\varepsilon')$-factors, and running couplings in $I(\varepsilon')$. The point is now that the N derivatives on ε will produce, when acting on $g(\varepsilon\varepsilon')$, precisely N positive powers of ε'. Thus, if N is sufficiently large, then the resulting positive powers will dominate the corresponding negative powers in the $Z(\varepsilon')$-factors, and we get the desired result. The generalization of this argument to arbitrary trees $\mathcal{T}$ can be done via an induction argument based upon formula (100), similar to the one given there.

To prove the Lorentz invariance of the W^a, let us now define, for each Lorentz transformation $\Lambda \in \mathrm{SO}(3,1)_0^{\uparrow}$ the diffeomorphism $f_\Lambda : \xi \mapsto \Lambda\xi$. It defines a causality and orientation preserving isometric embedding between the spacetimes (M, g) and $(M, f_\Lambda^* g = g_\Lambda)$ with the same orientations. Thus, using the local covariance property $f_\Lambda^* C^{\mathcal{T}}[g] = C^{\mathcal{T}}[g_\Lambda]$, and the transformation property of $f_\Lambda^* P^a = D^a{}_b(\Lambda) P^b$ under this diffeomorphism, it follows that

$$\sum_a P_a[\nabla^k R_{\alpha_1\alpha_2\alpha_3\alpha_4}(x_n), \xi_i^\nu]\, W^a(\Lambda\xi_1, \ldots, \Lambda\xi_{n-1}) \\ = \sum_{a,b} P_a[\nabla^k R_{\alpha_1\alpha_2\alpha_3\alpha_4}(x_n), \xi_i^\nu]\, D^a{}_b(\Lambda) W^b(\xi_1, \ldots, \xi_{n-1})\,. \quad (104)$$

However, since this holds for all metrics, eq. (95) follows.

Using the definition of W^a just given, item 4) immediately follows from the fact that C_I has an expansion of the form (89), and that $C_I^{\mathcal{T}} \sim_{\delta,\mathcal{T}} C_I$. □

5 Example

We now illustrate our general method for computing the OPE in curved spacetime by an example. Let us summarize again the steps needed in this computation.

1. Fix a desired accuracy, δ, of the OPE, and determine Δ as in Theorem 1.

2. Identify the retarded products in eq. (49) that are needed to compute the desired OPE coefficient to a given order in perturbation theory and a given accuracy δ, and determine them, using the methods of the papers [21, 22].

3. Perform a local "Wick expansion (27) of all retarded products. In places in (49) where two retarded products are multiplied, perform the products using Wick's theorem (10) (with ω_2 in that formula replaced by H). Apply the standard functionals (45) to the resulting expressions in the way indicated in (49).

This yields the desired OPE coefficient. If one is interested in the asymptotic behavior of the OPE coefficient (up to order δ in ε) under a rescaling of a point $\psi_{\mathcal{T}}(\varepsilon) : \vec{x} \mapsto \vec{x}(\varepsilon)$ associated with a given tree $\mathcal{T}$, perform the following step:

4. Take the Mellin-transform of $C_I(\vec{x}(\varepsilon))$ in ε as in eq. (90), and define $C_I^{\mathcal{T}}(\vec{x})$ to be the sum of its residue at the poles $\mathrm{i}\delta, \mathrm{i}(\delta-1), \ldots$, as in eq. (91). The result then automatically has the form of curvature terms times Minkowski distributions in the relative coordinates as described in item 2) of Theorem 2, and it is equivalent to C_I in the sense that $C_I^{\mathcal{T}} \sim_{\mathcal{T},\delta} C_I$.

As an illustration, we now apply this method to the determine the coefficient C_I in the triple product of operators

$$\phi(x_1)_I \phi(x_2)_I \phi(x_3)_I = \cdots + C(x_1, x_2, x_3)_I \, \phi(x_3)_I + \ldots \tag{105}$$

up to first order in perturbation theory in the interaction $I = \int_M \mathcal{L} \, d\mu = -\frac{1}{4!} \int_M \kappa \phi^4 \, d\mu$, and accuracy $\delta = 0$. We then discuss the scaling expansion as in item 2) of Theorem 2. As discussed above, we impose an infrared cutoff by taking $\kappa(x) = \kappa\chi(x)$ at intermediate steps, where χ is a smooth cutoff function, but the final answers will not depend on the choice of χ, see eq. (77). The different ways of scaling the 3 points together give rise to different limiting behavior $C_I^{\mathcal{T}}$ of C_I. We choose to investigate the most interesting case when all points are scaled together at the same rate, i.e., under the scaling map $\psi_{\mathcal{T}}(\varepsilon) : \vec{x} \to \vec{x}(\varepsilon)$, where $\mathcal{T} = \{\{1,2,3\},\{1\},\{2\},\{3\}\}$. If ξ_i are the Riemannian normal coordinates of the points x_i around x_3, this corresponds to $\vec{\xi}(\varepsilon) = (\varepsilon\xi_1, \varepsilon\xi_2, \varepsilon\xi_3)$ with $\xi_3 = 0$.

Consider first the zeroth order perturbative contribution. According to eq. (49), this is given for a general OPE coefficient of a triple product of operators by

$$C_{i_1 i_2 i_3}{}^j(x_1, x_2, x_3)_{(0)} = \Psi^j(\mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\mathcal{O}_{i_3}(x_3)) \,, \tag{106}$$

where the reference point for the functional Ψ^j (see eq. (45)) is x_3 throughout. We are interested in the case $\mathcal{O}_{i_1} = \mathcal{O}_{i_2} = \mathcal{O}_{i_3} = \mathcal{O}_j = \phi$. In order to determine the action of the functional Ψ^j in this case, we need to perform the local Wick expansion of the product $\phi(x_1)\phi(x_2)\phi(x_3)$, which is given by

$$\phi(x_1)\phi(x_2)\phi(x_3) = :\phi^{\otimes 3}:_{\mathrm{H}} (x_1, x_2, x_3) + H(x_1, x_2)\phi(x_3) + \text{ cyclic}(1,2,3) \tag{107}$$

Applying now the definition of the functional Ψ^j (with $\mathcal{O}_j$ chosen to be ϕ, and reference point x_3) gives

$$C(x_1, x_2, x_3)_{(0)} = H(x_1, x_2) + H(x_2, x_3) + H(x_1, x_3) \,, \tag{108}$$

for any δ.

In order to determine the representer $C_I^{\mathcal{T}}$ to zeroth order in perturbation theory for our choice $\delta = 0$ (see eq. (91)), we are instructed to compose the above result with the map $\psi_{\mathcal{T}}(\varepsilon)$, then take a Mellin transform in the variable ε, and then extract the residue at the poles $0, -\mathrm{i}, -2\mathrm{i}$ in the complex z-plane via the contour integral (91). As explained above, taking a Mellin transform and then extracting those residue is a way to compute the corresponding coefficients of $\varepsilon^0, \varepsilon^{-1}, \varepsilon^{-2}$ in the expansion (89) of the distribution $C_I \circ \psi_{\mathcal{T}}(\varepsilon)$ in ε. In the present simple example, it is easier to compute the coefficients directly from eq. (108), by using the corresponding short distance expansions [10] around x_3 of the quantities σ, v_n appearing in the Hadamard parametrix H. The result is

$$C^{\mathcal{T}}(x_1, x_2, x_3)_{(0)} = \frac{1}{2\pi^2} \sum_{i<j} \Big\{ \frac{1}{(\xi_i - \xi_j)^2} - \frac{1}{3} \frac{R_{\mu\nu\sigma\rho} \xi_i^\mu \xi_j^\nu \xi_i^\sigma \xi_j^\rho}{(\xi_i - \xi_j)^4} - \frac{1}{6} \frac{R_{\mu\nu}(\xi_i^\mu \xi_i^\nu + \xi_i^\mu \xi_j^\nu + \xi_j^\mu \xi_j^\nu)}{(\xi_i - \xi_j)^2} + \frac{1}{12} R \ln(\xi_i - \xi_j)^2 \Big\}, \quad (109)$$

where all curvature tensors are taken at x_3. By item 1) of Theorem 2, we know that $C_I \sim_{\mathcal{T},0} C_I^{\mathcal{T}}$.

We next consider the first order perturbative contribution to a general OPE coefficient for a general triple product. By formula (49) this is given by

$$C_{i_1 i_2 i_3}{}^j(x_1, x_2, x_3)_{(1)} = \mathrm{i} \int_M \Big[\Psi^j(\mathcal{O}_{i_1}(x_1) \mathcal{O}_{i_2}(x_2) \mathbf{R}_1(\mathcal{O}_{i_3}(x_3), \mathcal{L}(y))) + \text{cyclic}(1,2,3) - \sum_{[k] \leq \Delta} \Psi^j(\mathbf{R}_1(\mathcal{O}_k(x_3), \mathcal{L}(y)) \Psi^k(\mathcal{O}_{i_1}(x_1) \mathcal{O}_{i_2}(x_2) \mathcal{O}_{i_3}(x_3)) \Big] \chi(y)\, d\mu(y)\,. \quad (110)$$

We are again interested in the case $\mathcal{O}_{i_1} = \mathcal{O}_{i_2} = \mathcal{O}_{i_3} = \mathcal{O}_j = \phi$. The constant Δ depends on the desired precision of the OPE governed by δ, and is given in Thm. 1. For our choice $\delta = 0$, we have we have to choose $\Delta = 3$. In this case, it can be seen that only $\mathcal{O}_k = \phi^3$ will make a contribution. Thus, the required retarded products and their Wick expansion in eq. (110) are

$$\mathbf{R}_1(\phi^n(x), \phi^m(y)) = \sum_{k=0}^{4} \frac{n! m!}{(n-k)!(m-k)!}\, r_k(x,y)\ :\phi^{n-k}(x) \phi^{m-k}(y):_{\mathrm{H}} \quad (111)$$

for $n = 1, 3$ and $m = 4$. Consequently, we need to know r_k for $k = 1, 3$. The method [22] for defining the "renormalized" distribution gives $r_1 = \mathrm{i}\Delta_R$, where Δ_R is the retarded

propagator, and

$$
\begin{aligned}
r_3 &= -\frac{1}{32} v_0^3 A^2 \left(\Theta(-t) \frac{\ln(\sigma + \mathrm{i}0t)}{\sigma + \mathrm{i}0t} \right) \\
&+ \frac{3}{2} v_0^2 \sum_n v_{n+1} \frac{\sigma^n}{2^n n!} A \left(\Theta(-t) \frac{\ln^2(\sigma + \mathrm{i}0t) + \frac{1}{2}\ln(\sigma + \mathrm{i}0t)}{\sigma + i0t} \right) \\
&+ \Theta(-t) \sum_{m,n} v_{m+1} v_{n+1} \frac{\sigma^{n+m}}{2^{n+m} n! m!} \left(3v_0 \frac{\ln^2(\sigma + \mathrm{i}0t)}{\sigma + \mathrm{i}0t} + \sum_k v_{k+1} \frac{\sigma^k}{2^k k!} \ln^3(\sigma + \mathrm{i}0t) \right) - \text{h.c.} ,
\end{aligned}
\tag{112}
$$

where $t(x,y) = \tau(x) - \tau(y)$, for any time function $\tau : M \to \mathbb{R}$. The symbol A stands for the operator[9]

$$
A = \Box + (\nabla^\mu \ln D) \nabla_\mu , \tag{113}
$$

where D is the VanVleck determinant defined above in eq. (17). The desired OPE coefficient can now be obtained. The result is

$$
\begin{aligned}
C(x_1, x_2, x_3)_{(1)} = \mathrm{i}\kappa \int_M [& H(x_1, y) H(x_2, y) r_1(x_3, y) \\
& + \text{cyclic}(1,2,3) - r_3(x_3, y)] \chi(y) \, d\mu(y) ,
\end{aligned}
\tag{114}
$$

where H is the local Hadamard parametrix. To get the desired representer $C_I^{\mathcal{T}}$ provided by Theorem 2, we again use the definitions eqs. (90) and (91) corresponding to our choice $\delta = 0$. Thus, we must compose C_I with $\psi_{\mathcal{T}}(\varepsilon)$, take a Mellin transform, and extract the residue at $0, -\mathrm{i}, -2\mathrm{i}$ in the complex z-plane. To do the Mellin transform (90), we perform first a short distance expansion of the integrand in (114) around x_3, using the corresponding expansions [10] of the quantities σ, v_n present in H, r_i. This short distance expansion will lead to a sum of terms, each of which is a curvature polynomial at x_3, times a Minkowski distribution in ξ_1, ξ_2 and the Riemannian normal coordinates of y. We may also set $\chi = 1$, since the residue are independent of the particular choice of χ, as proven in Thm. 2. Thus, the computation of the Mellin-transform reduces to ordinary Minkowski integrals, times curvature polynomials at x_3. Only a finite number of these terms will give rise to poles at $0, -\mathrm{i}, -2\mathrm{i}$, and so all others can be discarded.

Let us consider in detail the pole $-2\mathrm{i}$. For this pole, the Minkowski space integrals that contribute can be reduced to the integral given in [31] using some standard "i0-identities" for distributions:

$$
\begin{aligned}
& C^{\mathcal{T}}(x_1, x_2, x_3)_{(1)} = \\
& \quad \frac{\kappa}{2^6 \pi^4} \frac{1}{\rho} \left[\mathrm{Cl}_2 \left(2 \arctan \frac{\rho}{\delta_1} \right) + \mathrm{Cl}_2 \left(2 \arctan \frac{\rho}{\delta_2} \right) + \mathrm{Cl}_2 \left(2 \arctan \frac{\rho}{\delta_3} \right) + \dots \right] ,
\end{aligned}
\tag{115}
$$

[9] The usefulness of this operator lies in the identity $A(\sigma^n) = 4n(n+1)\sigma^{n-1}$.

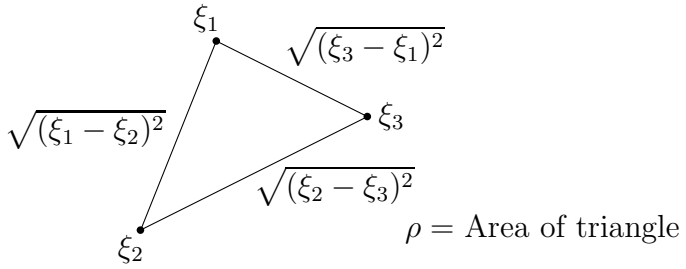
where the dots stand for the contributions from the poles $0, -\mathrm{i}$. Here, we have set

$$\rho = \sqrt{|\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1|} \tag{116}$$

as well as

$$\delta_1 = (\xi_2 - \xi_3)(\xi_3 - \xi_1), \quad \delta_2 = (\xi_1 - \xi_2)(\xi_2 - \xi_3), \quad \delta_3 = (\xi_3 - \xi_1)(\xi_1 - \xi_2), \tag{117}$$

and as before it is assumed that the points $\xi_i \in \mathbb{R}^4$ are pairwise spacelike, and $\xi_3 = 0$. The function Cl_2 is the Clausen function [30]. The arguments of these functions in the above expression are given by twice the angles of a triangle with sides of length $\sqrt{(\xi_1 - \xi_2)^2}$, $\sqrt{(\xi_2 - \xi_3)^2}$ and $\sqrt{(\xi_3 - \xi_1)^2}$, and ρ represents the area of that triangle, see the figure.



Thus, the result for $C_I^{\mathcal{T}}$ manifestly has the simple form claimed in the scaling expansion in Theorem 2. There are no curvature terms in the terms that we have displayed, but those arise from the other poles $0, -\mathrm{i}$. These contributions may be obtained in closed form using the Minkowski integrals of [8]. The end result is a sum of terms $\sum_a R_{\mu\nu\sigma\rho} W_a^{\mu\nu\sigma\rho}(\xi_1, \xi_2, \xi_3)$, where $W_a^{\mu\nu\sigma\rho}$ are Lorentz invariant distributions in the Riemannian normal coordinates ξ_i of x_i around x_3. However, the expressions for these distributions are rather complicated and will therefore be given elsewhere [26].

It should be clear that our method is not confined to the above example, but in principle only limited by the ability to perform complicated Minkowski integrals of Feynman type.

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